NAVAL RESEARCH LOGISTICS QUARTERLY

DECEMBER 1973
VOL. 20, NO. 4

OFFICE OF NAVAL RESEARCH

NAVSO P-1278

NAVAL RESEARCH LOGISTICS QUARTERLY

The Naval Research Logistics Quarterly is devoted to the dissemination of scientific information in logistics and will publish research and expository papers, including those in certain areas of mathematics, statistics, and economics, relevant to the over-all effort to improve the efficiency and effectiveness of logistics operations.

Information for Contributors is indicated on inside back cover.

The Naval Research Logistics Quarterly is published by the Office of Naval Research in the months of March, June, September, and December and can be purchased from the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 20402. Subscription Price: $10.00 a year in the U.S. and Canada, $12.50 elsewhere. Cost of individual issues may be obtained from the Superintendent of Documents.

Issuance of this periodical approved in accordance with Department of the Navy Publications and Printing Regulations, NAVEXOS P-35

GENERALIZED MULTICOMPONENT SYSTEMS UNDER CANNIBALIZATION

Murray Hochberg

Brooklyn College (CUNY)
Brooklyn, N.Y.

ABSTRACT

Using the theory of Hirsch, Meisner, and Boll, we study the consequences of interchanging parts within a generalized coherent structure. This procedure has been termed "cannibalization." The theory of cannibalization is extended to the case where each component can operate at several levels of partial performance and a representation theorem is derived, which expresses the state of a system as a function of the number of working parts at each level. The stochastic theory of these systems is then investigated.

INTRODUCTION

Coherent structures and similar multicomponent systems have been studied extensively [1, 3, 4]. In particular, Hirsch, Meisner, and Boll [3] have studied the effect of shifting operational parts from one location to another, a procedure called "cannibalization." The author uses the theory of Hirsch, Meisner, and Boll to study the consequences of interchanging parts within a generalized coherent structure. The theory of cannibalization is extended to the case where each component can operate at several levels of partial performance and the structure is permitted to take on several possible values of performance. The main result is a representation theorem, which expresses the state of a system as a function of the number of working parts at each level. The stochastic theory of these systems is studied and a formula is derived for the probability distribution of the cannibalized structure function.

1. ALGEBRAIC THEORY

We introduce the abstract set $\Lambda=\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, whose points represent the loci of the structure. We assume that at each moment of time each locus is in one of the following k possible states:

(a) It is in the "best" operational state, in which case we associate to the locus the value a_1, $a_1=1$.

(b) It assumes one of the $k-2$ "intermediate" operational values, $\{a_2, a_3, \ldots, a_{k-1}\}$ with

$$1=a_1>a_2>a_3>\ldots>a_{k-1}>0.$$

(c) It assumes the value a_k, $a_k=0$, which signifies that the locus fails to contain an operational part. Thus, at any fixed moment of time, the states of all the loci are described by a mapping

$$v:\Lambda\rightarrow\{a_1, a_2, \ldots, a_k\},$$

whose value at λ_i,

$$x_i=v(\lambda_i),$$

denotes the state of λ_i. We will denote the totality of the possible states of the loci by K^n, where

$$K^n = \{(z_1, z_2, \ldots, z_n) : z_i = a_j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, k\}.$$

Each point $v \in K^n$ is called a *locus-vector state.*

We shall assume that two parts O_1 and O_2 can be interchanged if and only if the following two conditions are satisfied:

(1.1) They are capable of functioning in precisely the same set of loci.

(1.2) When O_1 is installed and operating at a given level in a given locus λ, its "contribution" to the "level of performance" of the structure is precisely the same as that of O_2, when O_2 is installed and operating at the same given level in λ. However, the contribution of a given part to the level of performance of the structure may depend on the locus λ in which the part is installed.

We shall make the assumption that the level of performance of a structure at a given moment of time is determined solely by the locations and states of all of its parts at that moment. Assumptions (1.1) and (1.2) then imply that the level of performance of the structure is determined, at any given instant, by the states of the loci. We assume that some measure of performance has been selected, and we denote the set of possible performance levels by $S = \{0, 1, \ldots, M\}$. We interpret the state "0" as total failure and the state "M" as perfect performance, and we call each possible level a *system state.*

We define a partial order in K^n by setting $y \leq x$ if and only if $(y)_i \leq (x)_i$, $i = 1, 2, \ldots, n$, where $(x)_i$ denotes the ith component of x.

We assume that we are given a *structure function*

$$\phi : K^n \to S,$$

whose value at the vertex $v \in K^n$ denotes the state of the system when the loci are in the states described by the vector v. A function $\phi : K^n \to S$ is said to be a *monotone structure function* if

(a) $$v \leq v' \text{ implies that } \phi(v) \leq \phi(v')$$

and

(b) $$\phi(\mathbf{0}) = 0, \quad \phi(\mathbf{1}) = M, \text{ where } \mathbf{0} = (0, 0, \ldots, 0) \text{ and } \mathbf{1} = (1, 1, \ldots, 1).$$

If $k = 2$ and $M = 1$, a monotone structure function is called a *coherent structure.* Without further explicit mention, all the structure functions considered in the sequel will be assumed to be monotone.

The set of parts satisfying (1.1) and (1.2) defines a part type. To make precise our assumption about interchangeability, we introduce the set

$$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_N\},$$

where $\gamma_1, \gamma_2, \ldots, \gamma_N$ represent the part types in the structure. Let 2^Λ denote the collection of all possible subsets of Λ. We suppose that we are given a mapping

$$Q : \Gamma \to 2^\Lambda,$$

whose value $Q(\gamma_i)$ represents the set of all loci in which parts of type γ_i are installed initially; then, under our restrictions of interchangeability, interchanges are permitted within $Q(\gamma_i)$, but not between a locus $\lambda \in Q(\gamma_i)$ and a locus $\lambda' \notin Q(\gamma_i)$. We postulate in addition that

$$Q(\gamma_i) \neq \phi, \qquad i = 1, 2, \ldots, N,$$

(1.3)

$$Q(\gamma_i) \cap Q(\gamma_j) = \phi, \qquad \text{if } i \neq j$$

and

$$\bigcup_{i=1}^{N} Q(\gamma_i) = \Lambda.$$

These properties express, respectively, the facts that each part type is associated with at least one locus, no locus has more than one part type associated with it, and each locus has some part type associated with it. Thus for each integer i, $1 \leq i \leq n$, there is a unique j, say $j = \delta(i)$, that satisfies the relation $\lambda_i \in Q(\gamma_j)$. The function δ identifies the part type used at each locus.

Definition. Given a set A, the *indicator function*, I_A, is defined by

$$I_A(v) = \begin{cases} 1, & \text{if } v \in A \\ 0, & \text{if } v \notin A. \end{cases}$$

Definition. For each integer j, $1 \leq j \leq N$, we set

$$w_j = (w_{j1}, w_{j2}, \ldots, w_{j,k-1}),$$

where

$$w_{jq} : K^n \to \{0, 1, \ldots, n\}$$

is the map defined by

$$w_{jq}(v) = \sum_{\{i : \lambda_i \in Q(\gamma_j)\}} I_{\{(v)_i = a_q\}}(v), \qquad q = 1, 2, \ldots, k-1.$$

The map w_{jq} can be interpreted as the number of operational parts of type γ_j operating at level a_q, $q = 1, 2, \ldots, k-1$.

The following properties of w_j are immediate:

(a) $$0 \leq \sum_{q=1}^{k-1} w_{jq}(v) \leq |Q(\gamma_j)|.$$

(b) $$\sum_{q=1}^{k-1} w_{jq}(v) = |Q(\gamma_j)|,$$

if and only if all $\lambda_i \in Q(\gamma_j)$ are operating at some positive level of performance.

(c) $$w_{ji}(v) = |Q(\gamma_j)|$$

if and only if $(v)_i = 1$ for all i such that $\lambda_i \epsilon Q(\gamma_j)$.

Definition. Two vertices $v \epsilon K^n$ and $v' \epsilon K^n$ will be called *w_j-equivalent*, in symbols,

$$v \overset{w_j}{\sim} v'$$

if and only if

$$w_j(v) = w_j(v').$$

Definition. The vertex v is said to be *equivalent* to the vertex v', in symbols,

$$v \sim v',$$

if and only if for each integer j, $1 \leqslant j \leqslant N$, we have

$$v \overset{w_j}{\sim} v'.$$

Clearly, "$\overset{w_j}{\sim}$" and "$\sim$" are equivalence relations, and we will denote the respective equivalence classes by $[v]_{w_j}$ and $[v]$. The class $[v]_{w_j}$ consists of all locus-vector states, v', for which the number of operational parts of type γ_j operating at level a_i, $i = 1, 2, \ldots, k-1$, is the same for both v and v'. The class $[v]$ consists of all locus-vector states, v', for which the number of operational parts of type γ_j, $1 \leqslant j \leqslant N$, operating at level a_i, $i = 1, 2, \ldots, k-1$, is the same for both v and v'. Since we have postulated that only parts of the same type can be interchanged, and since an interchange clearly does not affect the number of working parts of any part type at any level, the operation of interchanging parts leads from a vertex v to a vertex v' in $[v]$. Guided by this we introduce the following definition:

Definition. A *cannibalization* is any transformation

$$T : K^n \to K^n$$

such that for all $v \epsilon K^n$,

$$Tv \epsilon [v].$$

We denote the class of all cannibalizations by $\mathscr{T}$.

Given a cannibalization T, we define the *cannibalized structure function*, ϕ^T, by

$$\phi^T(v) = \phi T(v) = \phi(Tv).$$

A cannibalization T is said to be *admissible* if

(1.4) $$\phi^T \geqslant \phi^{T'} \text{ for all } T' \epsilon \mathscr{T},$$

which expresses the fact that an admissible cannibalization is uniformly as good as any other cannibalization. Plainly, T is admissible if and only if

$$\phi^T(v) = \max_{v' \in [v]} \phi(v'), \qquad v \in K^n.$$

Since there may be many points in $[v]$ at which ϕ assumes the value $\max_{v' \in [v]} \phi(v')$, in general there are many admissible cannibalizations. We will denote the class of admissible cannibalizations by $\mathcal{T}^*$. It is clear that each $T \in \mathcal{T}^*$ induces precisely the same cannibalized structure function, which we denote by ϕ^*. Clearly, ϕ^* is constant on each equivalence class $[v]$. Moreover, $\phi \leqslant \phi^*$, with equality if and only if $u \sim v$ implies $\phi(u) = \phi(v)$.

Henceforth, we consider only admissible cannibalizations.

Definition. Let $\bar{u}$ be any point in $[u]$ such that for all $u' \in [u]$,

$$\phi(\bar{u}) \geqslant \phi(u').$$

We call such a point a *maximum point of the restriction of* ϕ *to* $[u]$. This restriction is denoted by $\phi|[u]$.

THEOREM 1.1: If ϕ is a monotone structure function, then ϕ^* is also a monotone structure function.

PROOF: Let $u = (i_1, i_2, \ldots, i_n)$ and $v = (k_1, k_2, \ldots, k_n)$ be two vertices in K^n, such that $u \leqslant v$. Let $\bar{u}$ be a maximum point of $\phi|[u]$. We will construct a vertex $\bar{v} \in [v]$ such that $\bar{u} \leqslant \bar{v}$. Since $\bar{u} \in [u]$,

$$\bar{u} = (i_{q_1}, i_{q_2}, \ldots, i_{q_n}),$$

where $(i_{q_1}, i_{q_2}, \ldots, i_{q_n})$ is a permutation of $(i_1, i_2, \ldots, i_n)$. We now define

$$\bar{v} = (k_{q_1}, k_{q_2}, \ldots, k_{q_n}).$$

Since

$$i_j \leqslant k_j, \qquad j = 1, 2, \ldots, n,$$

we have

$$i_{q_j} \leqslant k_{q_j}, \qquad j = 1, 2, \ldots, n.$$

Thus,

$$\bar{u} \leqslant \bar{v}, \text{ with } \bar{v} \in [v].$$

Then

$$\phi^*(u) = \phi(\bar{u}) \leqslant \phi(\bar{v}) \leqslant \phi^*(v).$$

We now introduce concepts that enable us to measure the extent to which the performance of the system depends individually on each part type.

Let $\Pi_i : K^n \to K^n$ be the mapping defined by

$$(\Pi_i u)_j = \begin{cases} (u)_j, & \text{if } \lambda_j \in Q(\gamma_i) \\ 1, & \text{otherwise, } u \in K^n. \end{cases}$$

The effect of Π_i on a locus-vector state is to transform it into one in which all the loci occupied by parts other than type γ_i are operational in state 1 and in which states of the loci corresponding to part type γ_i are left unchanged.

The following properties of Π_i are immediate:

(1.5)
(a) Π_i is nondecreasing; i.e., $\Pi_i v \geq v$.
(b) Π_i is order preserving; i.e., if $u \geq v$, then $\Pi_i u \geq \Pi_i v$.
(c) $u \overset{w_i}{\sim} v$ implies that $\Pi_i u \sim \Pi_i v$.

Using the definition of a cannibalization T and (1.5c) we conclude that if $u \overset{w_i}{\sim} v$, then $T\Pi_i u \sim \Pi_i v$. In particular, we have

$$T\Pi_i v \sim \Pi_i v \overset{w_i}{\sim} v. \tag{1.6}$$

For each integer i, $1 \leq i \leq N$, we define the *structure function relative to* i, ϕ_i, by

$$\phi_i = \phi\Pi_i.$$

The function ϕ_i describes how the structure would perform without cannibalization if an infinite number of spares for all parts other than γ_i were available. Since Π_i is nondecreasing and order preserving,

$$\phi_i \geq \phi$$

and, if ϕ is monotone, so is ϕ_i.

We define the *cannibalized structure function relative to* i, ϕ_i^*, by

$$\phi_i^*(v) = \phi^*(\Pi_i v) = \phi(T\Pi_i v).$$

The function ϕ_i^* is constant on each equivalence class $[v]_{w_i}$; i.e., there is a function [1]

$$f_i : \{0, 1, \ldots, |Q(\gamma_i)|\}^{k-1} \to S$$

such that for all $v \in K^n$,

$$\phi_i^*(v) = f_i w_i(v).$$

[1] We use the standard notation, $A^k = A_1 \times A_2 \times \ldots \times A_k$, where $A_i = A$, $i = 1, 2, \ldots, k$.

For if $v \overset{w_i}{\sim} v'$, then by (1.5), $\Pi_i v \sim \Pi_i v'$, and therefore

$$\phi_i^*(v) = \phi^*(\Pi_i v) = \phi^*(\Pi_i v') = \phi_i^*(v').$$

THEOREM 1.2: If ϕ^* is a monotone structure function, then f_i is nondecreasing in each variable.

PROOF: Let $(x_1, x_2, \ldots, x_{k-1}) < (y_1, y_2, \ldots, y_{k-1})$. Let u and v be two vertices such that $w_i(u) = (x_1, x_2, \ldots, x_{k-1})$ and $w_i(v) = (y_1, y_2, \ldots, y_{k-1})$. As in the proof of Theorem 1.1, we can construct $v' \epsilon [v]_{w_i}$ such that $u < v'$. Since ϕ^* and Π_i are nondecreasing, we conclude that

$$f_i(x_1, x_2, \ldots, x_{k-1}) = f_i w_i(u) = \phi_i^*(u) \leq \phi_i^*(v')$$

$$= \phi_i^*(v) = f_i w_i(v) = f_i(y_1, y_2, \ldots, y_{k-1}).$$

DEFINITION: A structure function ϕ is said to satisfy the *minimum condition* if

$$\phi = \min_{1 \leq j \leq N} \phi_j.$$

Intuitively, one can interpret the minimum condition as asserting that the value that the structure function assumes at any vertex is determined by one particular part type, in the sense that if all other part types were to be made fully operational, the value of the structure function remains unchanged. The responsible part type may depend on the vertex and need not be unique.

THEOREM 1.3: If $\phi = \min_{1 \leq j \leq N} \phi_j$, then $\phi^* = \min_{1 \leq j \leq N} \phi_j^*$.

THEOREM 1.4: Let ϕ be any structure function that induces ϕ^*. The relation

$$\phi^* = \min_{1 \leq j \leq N} \phi_j^*$$

holds if and only if to each maximum point $\bar{v}$ of $\phi | [v]$ there corresponds an integer i_0, depending on $\bar{v}$, such that $\Pi_{i_0} \bar{v}$ is a maximum point of $\phi | [\Pi_{i_0} \bar{v}]$ and

$$\phi(\bar{v}) = \phi(\Pi_{i_0} \bar{v}).$$

THEOREM 1.5: If $\phi^* = \min_{1 \leq j \leq N} \phi_j^*$, then

$$\min_{1 \leq j \leq N} \phi T \Pi_j = \min_{1 \leq j \leq N} \phi \Pi_j T.$$

We omit the proofs of Theorems 1.3, 1.4, and 1.5, since the proofs given in [3] can be easily extended to the more general model considered here.

We are now in a position to derive an algebraic representation of the cannibalized structure function ϕ^*. We recall that

$$\phi_i^* = f_i w_i,$$

where the function f_i is nondecreasing in each variable. Let $A_{i,q}$, $1 \leq i \leq N$, $0 \leq q \leq M+1$, denote the set of $(k-1)$-tuples in the domain of f_i, for which f_i takes a value at least as large as q. Since f_i is nondecreasing in each variable, we have

$$A_{i,0} \supseteq A_{i,1} \supseteq \ldots \supseteq A_{i,M} \supseteq A_{i,M+1}, \qquad 1 \leq i \leq N.$$

Since $\phi_i^* \leq M$, we conclude that $A_{i,M+1} = \phi$. Moreover, since $\phi_i^*(1, 1, \ldots, 1) = M$, we have $A_{i,M} \neq \phi$. Thus, $A_{i,q} \neq \phi$ if $q \leq M$.

We say that x is a *minimal point* of $A_{i,q}$ if $x \in A_{i,q}$ and there does not exist any $y \in A_{i,q}$, with $y < x$. We define the set $n_i(q)$, $0 \leq q \leq M$, to be the set of minimal points of $A_{i,q}$. Moreover, we set

$$n_i(M+1) = \{(|Q(\gamma_i)|+1, |Q\gamma_i)|+1, \ldots, |Q(\gamma_i)|+1)\}.$$

DEFINITION: Given two collections of vectors $\mathscr{A}$ and $\mathscr{B}$, we say that $\mathscr{A} \geq \mathscr{B}$ (or $\mathscr{B} \leq \mathscr{A}$) if and only if for every vector $y \in \mathscr{A}$ there exists at least one vector $z \in \mathscr{B}$ such that $y \geq z$.

LEMMA 1.1: The relation "$\geq$" as defined above is a partial order on the set $n_i(q)$.

PROOF: We must verify the three properties of a partial ordering:

(a) $\mathscr{A} \geq \mathscr{A}$.
(b) If $\mathscr{A} \geq \mathscr{B}$ and $\mathscr{B} \geq \mathscr{A}$, then $\mathscr{A} = \mathscr{B}$.
(c) If $\mathscr{A} \geq \mathscr{B}$ and $\mathscr{B} \geq \mathscr{C}$, then $\mathscr{A} \geq \mathscr{C}$.

Properties (a) and (c) are obvious, and thus all we need to verify is property (b). Let $y \in \mathscr{A}$. Then since $\mathscr{A} \geq \mathscr{B}$, there exists $z \in \mathscr{B}$ such that $y \geq z$. Since $\mathscr{B} \geq \mathscr{A}$, there exists $y' \in \mathscr{A}$ such that $z \geq y'$. Thus,

$$y \geq z \geq y'. \tag{1.7}$$

Since y, $y' \in n_i(q)$, it is impossible that $y > y'$. Thus we have equality in (1.7), and $y = z$. By symmetry it follows that $\mathscr{A} = \mathscr{B}$.

We define

$$w_i(v) \geq n_i(q)$$

to mean

$$\{w_i(v)\} \geq n_i(q).$$

THEOREM 1.6: For all integers i, q, $1 \leq i \leq N$, $0 \leq q \leq M+1$, and $v \in K^n$,

$$f_i w_i(v) \geq q$$

if and only if

$$w_i(v) \geq n_i(q).$$

PROOF: For $q = M+1$ the theorem is true since $w_i(v) \leq (|Q(\gamma_i)|, |Q(\gamma_i)|, \ldots, |Q(\gamma_i)|)$ and $f_i w_i(v) \leq M$. Assume now that $f_i w_i(v) \geq q$, $0 \leq q \leq M$. Then $w_i(v) \in A_{i,q}$. If $w_i(v) \in n_i(q)$, then surely

$w_i(v) \geq n_i(q)$. Hence, it will suffice to examine the case, $w_i(v) \notin n_i(q)$. If $w_i(v) \notin n_i(q)$, there exists a $(k-1)$-tuple $x \in n_i(q)$ such that $w_i(v) > x$. Then $w_i(v) \geq n_i(q)$.

Now assume $w_i(v) \geq n_i(q)$. Since f_i is nondecreasing and $n_i(q) \subseteq A_{i,q}$, we have that $f_i w_i(v) \geq q$.

Theorem 1.6 shows that the level of the cannibalized structure relative to i is at least as large as q if and only if there are "at least" $n_i(q)$ parts of type γ_i operating, where "at least" is to be interpreted in the "$\geq$" sense.

THEOREM 1.7: Let $q < q'$. Then $n_i(q) \leq n_i(q')$.

PROOF: First consider $q < q' \leq M$. As indicated previously, $A_{i,q'} \subseteq A_{i,q}$. In particular, $n_i(q') \subseteq A_{i,q}$. Let x' be an arbitrary $(k-1)$-tuple in $n_i(q')$. If for every $x' \in n_i(q')$, x' is also in $n_i(q)$, then $n_i(q) \leq n_i(q')$. If there exists an $x' \in n_i(q')$, $x' \notin n_i(q)$, there must exist $x \in n_i(q)$ with $x < x'$. Then $n_i(q) \leq n_i(q')$.

For $q' = M+1$ the theorem is true since $y \in A_{i^\circ q}$, $0 \leq q \leq M$, implies that all the components of y are less than or equal to $|Q(\gamma_i)|$.

We derive finally a useful algebraic representation of the structure function.

THEOREM 1.8: (Representation Theorem): If $\phi^* = \min_{1 \leq i \leq N} \phi_i^*$, then

$$\phi^* = \sum_{k=1}^{M} \prod_{i=1}^{N} I_{\{w_i \geq n_i(k)\}} = \sum_{k=1}^{M} I_{\bigcap_{i=1}^{N} \{w_i \geq n_i(k)\}}$$

PROOF: Let $v \in K^n$ be arbitrary and let $\phi^*(v) = k_v$. Then by hypothesis there exists an integer i_0, $1 \leq i_0 \leq N$, such that

$$\min_{1 \leq i \leq N} \phi_i^*(v) = \phi_{i_0}^*(v) = f_{i_0} w_{i_0}(v) = k_v.$$

We first consider the case, $k \geq k_v + 1$. Since $\phi_{i_0}^*(v) \geq k_v$, Theorem 1.6 implies that $w_{i_0}(v) \geq n_{i_0}(k_v)$. Moreover, since $\phi_{i_0}^*(v) < k_v + 1$, it follows from Theorem 1.6 that

$$w_{i_0}(v) \not\geq n_{i_0}(k_v + 1). \tag{1.8}$$

Since $k \geq k_v + 1$, we conclude from Theorem 1.7 that

$$n_{i_0}(k_v + 1) \leq n_{i_0}(k). \tag{1.9}$$

Relations (1.8) and (1.9) imply that

$$w_{i_0}(v) \not\geq n_{i_0}(k).$$

Thus, for $k \geq k_v + 1$,

$$I_{\{w_{i_0}(v) \geq n_{i_0}(k)\}}(v) = 0,$$

which implies that

$$\prod_{i=1}^{N} I_{\{w_i \geq n_i(k)\}}(v) = 0. \tag{1.10}$$

Now suppose that $k \leq k_v$. Since $\phi^* = \min_{1 \leq i \leq N} \phi_i^*$, and since $\phi^*(v) = k_v$, we have $k_v \leq \phi_i^*(v)$ for all integers i, $1 \leq i \leq N$. Using Theorems 1.6 and 1.7, we conclude that

$$n_i(k) \leq n_i(k_v) \leq w_i(v), \qquad 1 \leq i \leq N.$$

Thus for $k \leq k_v$, we have

$$\prod_{i=1}^{N} I_{\{w_i \geq n_i(k)\}}(v) = 1. \tag{1.11}$$

By combining (1.10) and (1.11),

$$\sum_{k=1}^{M} \prod_{i=1}^{N} I_{\{w_i \geq n_i(k)\}}(v) = \sum_{k=1}^{k_v} \prod_{i=1}^{N} I_{\{w_i \geq n_i(k)\}}(v) + \sum_{k=k_v+1}^{M} \prod_{i=1}^{N} I_{\{w_i \geq n_i(k)\}}(v)$$

$$= k_v + 0 = k_v = \phi^*(v).$$

Since, for any collection of sets $\{A_i\}_{i=1}^{N}$,

$$I_{\bigcap_{i=1}^{N} A_i} = \prod_{i=1}^{N} I_{A_i},$$

the proof is complete.

2. STOCHASTIC THEORY

We now introduce into our model random variables that make it possible to study mathematically the role of chance in the life history of a system subject to cannibalization. Our main goal is to determine at a given moment of time the probability that the system is operating at level k, $k = 0, 1, \ldots, M$.

We consider a system governed by a monotone structure function ϕ and assume at time $t = 0$ there are s_i spares available of part type γ_i, $i = 1, 2, \ldots, N$. There are many possible service policies prior to stockout of a given part type (i.e., before spares have been exhausted). The policy we consider is the one in which a spare is installed only when a part has degenerated to the "total-failure" stage; i.e., its locus has operational value 0. After the supply of spares of a given part type has been exhausted, a failure of that type is serviced by performing an admissible cannibalization.† For mathematical simplicity, we assume that cannibalizations and replacements of failed parts are performed instantaneously.

---

†It should be noted that the results obtained in this paper remain valid if admissible cannibalizations are also performed following each partial part failure, both before and after stockout.

We define the stochastic process†

$$\{V^T(t)=(V_1^{(T)}(t),\quad V_2^{(T)}(t),\ \ldots,V_n^{(T)}(t)),t\geq 0\},$$

where for each fixed t, $V_i^{(T)}$ is a random variable whose possible values are a_j, $j=1, 2, \ldots, k$. We interpret $V_i^{(T)}(t)$ as the state of the part in locus λ_i at time t if the cannibalization T is used. We set

$$\Phi^*(t)=\phi^*V^T(t)$$

and

$$W_i^T(t)=w_iV^T(t).$$

Clearly, $\Phi^*(t)$ represents the state of the cannibalized system at time t, and $W_i^T(t)$ represents the random number of operating parts of type γ_i at time t.

Assuming that ϕ^* satisfies the minimum condition, it is easily shown from the representation theorem that

$$\Phi^*(t)=\sum_{k=1}^{M}\prod_{i=1}^{N}I_{\{W^T{}_i(t)\geq n_i(k)\}} \tag{2.1}$$

$$=\sum_{k=1}^{M}I_{\bigcap_{i=1}^{N}\{W_i^T(t)\geq n_i(k)\}}.$$

As shown in [3], the probability distribution of $\Phi^*(t)$ may depend upon the particular admissible cannibalization used. In order to eliminate the dependence of the probability distribution of $\Phi^*(t)$ on $T\epsilon\mathcal{T}$, we postulate the following:

(a) The failure rate at each instant of a given part type γ_i, $i=1, 2, \ldots, N$, does not depend on the particular locus in $Q(\gamma_i)$ in which the part is installed, nor on the particular sequence of loci through which it has passed.

(b) Parts operate independently; i.e., the lifetime of a given part is not related to the lifetimes of any other parts.

Postulate (a) implies that the joint distribution of $(W_1^T(t), W_2^T(t), \ldots, W_N^T(t))$ doesn't depend on the particular cannibalization $T\epsilon\mathcal{T}$ employed. Postulate (b) implies that the N stochastic processes

$$\{W_1(t), t\geq 0\}, \{W_2(t), t\geq 0\}, \ldots, \{W_N(t), t\geq 0\}$$

are mutually independent; the superscript T is omitted because of postulate (a).

†Although the stochastic process $\{V^T(t), t\geq 0\}$ and the other processes discussed in this section are not explicitly constructed, it will be plain that in an appropriate model all the functions that we consider are measurable.

Hence, using (2.1), we conclude that the expected state of the cannibalized system at time t is

$$E[\Phi^*(t)] = \sum_{k=1}^{M} \prod_{i=1}^{N} E[I_{\{W_i(t) \geq n_i(k)\}}] = \sum_{k=1}^{M} \prod_{i=1}^{N} P\{W_i(t) \geq n_i(k)\}, \tag{2.2}$$

where P is the underlying probability measure defined on 2^{K^n}.

Setting

$$I_k = I_{\bigcap_{i=1}^{N} \{W_i(t) \geq n_i(k)\}}, \qquad k = 1, 2, \ldots, M,$$

and recalling that

$$n_i(1) \leq n_i(2) \leq \ldots \leq n_i(M),$$

we conclude that

$$I_1 \geq I_2 \geq \ldots \geq I_M. \tag{2.3}$$

Hence, it follows from (2.1) and (2.3) that

$$\{\Phi^*(t) \geq j\} = \left\{\sum_{k=1}^{M} I_k \geq j\right\} = \{I_j = 1\}.$$

Therefore,

$$\begin{aligned} P\{\Phi^*(t) \geq j\} &= P\{I_j = 1\} \\ &= P\left(\bigcap_{i=1}^{N} \{W_i(t) \geq n_i(j)\}\right) \\ &= \prod_{i=1}^{N} P\{W_i(t) \geq n_i(j)\}, \end{aligned} \tag{2.4}$$

where the last step follows from postulate (b).

Thus, in order to evaluate the probability distribution and the expected value of $\Phi^*(t)$, it is necessary to compute $P\{W_i(t) \geq n_i(j)\}$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. The remainder of this section is devoted to this calculation. Our calculations will apply to an arbitrary, but fixed part type γ_i, and accordingly we shall omit the subscript i from all the relevant expressions.

We begin with the postulate that the lifetimes of parts of a given type are identically distributed random variables. Moreover, postulates (a) and (b) imply that the probability distribution of $W(t)$ is independent of the particular admissible cannibalization used and, indeed, of whether or not cannibalization is practiced. Thus to determine the probability distribution of $W(t)$, it suffices to consider a model in which at time $t = 0$, $|Q|$ parts of type γ are installed and functioning at level 1, s spares are available, and cannibalization is not practiced.

For simplicity, we treat the case in which there are only three possible operational states for each locus: $a_1=1$, a_2, and $a_3=0$, with $0<a_2<1$. Thus, in order to use (2.2) and (2.4), we must evaluate

$$P\{\xi_1(t)=j, \quad \xi_2(t)=k\}, \tag{2.5}$$

where $W(t)=(\xi_1(t), \xi_2(t))$, and $\xi_\nu(t)$ denotes the number of operational parts at level a_ν at time t.

We consider first the case when $s=0$. Let $O_1, O_2, \ldots, O_{|Q|}$ denote the $|Q|$ installed parts, and let $Z_\nu(O_j)$ be the random variable representing the lifetime in state a_ν of part O_j, $\nu=1, 2, j=1, 2, \ldots, |Q|$. Since the performances of parts in different loci are independent of each other, $Z_1(O_1)$, $Z_1(O_2), \ldots, Z_1(O_{|Q|})$ are independent, identically distributed random variables with a common distribution function, say F. Similarly, $Z_2(O_1), Z_2(O_2), \ldots, Z_2(O_{|Q|})$ have a common distribution function, say G, and, by assumption,

$$Z_1=(Z_1(O_1), Z_1(O_2), \ldots, Z_1(O_{|Q|}))$$

and

$$Z_2=(Z_2(O_1), Z_2(O_2), \ldots, Z_2(O_{|Q|}))$$

are independent random vectors. Moreover, we assume that $Z_1(O_j)$ and $Z_2(O_j)$ are independent random variables, $j=1, 2, \ldots, |Q|$.

We define $A(t)$, $B(t)$, and $C(t)$, respectively, by

$$A(t)=P\{Z_1(O_j)+Z_2(O_j)<t\}, \tag{2.6}$$

$$B(t)=P\{Z_1(O_j)<t, Z_1(O_j)+Z_2(O_j)\geq t\}, \tag{2.7}$$

and

$$C(t)=P\{Z_1(O_j)\geq t\}; \tag{2.8}$$

by assumption $A(t)$, $B(t)$, and $C(t)$ do not depend on j. Clearly, $A(t)$ is the probability that a part installed at time $t=0$ will pass to state 0 before time t. We call this event a *complete failure*. $B(t)$ represents the probability that a part installed at time $t=0$ will pass to state a_2 before time t and that it will remain in state a_2 at least until time t. We call this event a *partial failure*. $C(t)$ is the probability that a part installed at time $t=0$ will remain in state 1 at least until time t. Plainly,

$$A(t)+B(t)+C(t)=1. \tag{2.9}$$

Moreover,

$$A(t)=F*G(t)=\int_0^t F(t-x)\,dG(x), \quad t\geq 0. \tag{2.10}$$

Similarly,

$$B(t) = \int_0^t dF(u) \int_{t-u}^{\infty} dG(v) \tag{2.11}$$

$$= F(t) - F*G(t), \qquad t \geq 0.$$

Clearly,

$$C(t) = 1 - F(t), \qquad t \geq 0. \tag{2.12}$$

Using (2.10), (2.11) and (2.12), we conclude that for $s = 0$,

$$P\{\xi_1(t) = j, \xi_2(t) = k\} = \binom{|Q|}{j, k, |Q| - j - k} [1 - F(t)]^j [F(t) - F*G(t)]^k [F*G(t)]^{|Q|-j-k}, t \geq 0. \tag{2.13}$$

Suppose next that $s > 0$. We shall not treat this case in full generality. Instead, we assume that the random variable $Z_\nu(O_j)$, $\nu = 1, 2$, $j = 1, 2, \ldots, Q$, has an exponential distribution with parameter $\lambda_\nu > 0$; thus,

$$F(x) = P\{Z_1(O_j) < x\} = \begin{cases} 1 - e^{-\lambda_1 x}, & x \geq 0 \\ 0, & x < 0, \end{cases}$$

and

$$G(x) = P\{Z_2(O_j) < x\} = \begin{cases} 1 - e^{-\lambda_2 x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Let $Z_\nu(O_{|Q|+k})$, $\nu = 1, 2$, be a random variable denoting the lifetime in state a_ν of the kth spare to be installed, $k = 1, 2, \ldots, s$. Since the spares and the originally installed parts are of the same type, our assumptions imply that for $\nu = 1, 2$, $Z_\nu(O_1), Z_\nu(O_2), \ldots, Z_\nu(O_{|Q|}), \ldots, Z_\nu(O_{|Q|+s})$ are independent and exponentially distributed with parameter λ_ν.

Under these assumptions, (2.10), (2.11) and (2.12) become, respectively,

$$A(t) = \int_0^t [1 - e^{-\lambda_1 (t-x)}] \lambda_2 e^{-\lambda_2 x} dx \tag{2.14}$$

$$= 1 - e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_1 - \lambda_2} [e^{-\lambda_2 t} - e^{-\lambda_1 t}], \qquad t \geq 0,$$

$$B(t) = \int_0^t \lambda_1 e^{-\lambda_1 u} du \int_{t-u}^{\infty} \lambda_2 e^{-\lambda_2 v} dv \tag{2.15}$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_2} [e^{-\lambda_2 t} - e^{-\lambda_1 t}], \qquad t \geq 0,$$

and

$$C(t) = 1 - [1 - e^{-\lambda_1 t}] = e^{-\lambda_1 t}, \qquad t \geqslant 0. \tag{2.16}$$

It is convenient to consider next the hypothetical situation in which there are available an infinite number of spares of the part type under consideration. In this case, whenever a part fails completely, it is immediately replaced by a new part, which is in state 1. Thus, at any moment of time, each of the $|Q|$ loci are either in state 1 or state a_2.

Choose any locus and let $\mu_\nu(t)$ denote the probability that, at time t, the component in use in that particular locus is in state a_ν, $\nu = 1, 2$. Since the lifetime in state 1 is exponential with parameter λ_1, it follows from standard arguments that the conditional probability that at time $t + h$ a part in a given locus is in state a_2 given that at time t it was in state 1, is $\lambda_1 h + o(h)$. Similarly, the conditional probability that at time $t + h$ a part in a given locus is in state 1 given that at time t the part in this locus was in state a_2 is $\lambda_2 h + o(h)$. Moreover, the probability that more than one passage occurs in a given locus in the time interval $(t, t + h)$ is $o(h)$, and the process describing the state of the locus at each moment of time is Markovian. The preceding observations imply that

$$\mu_1(t + h) = [1 - \lambda_1 h]\mu_1(t) + \lambda_2 h \mu_2(t) + o(h),$$

or

$$\frac{\mu_1(t + h) - \mu_1(t)}{h} = -\lambda_1 \mu_1(t) + \lambda_2 \mu_2(t) + \frac{o(h)}{h}.$$

Letting $h \to 0$, and using the fact that

$$\mu_2(t) = 1 - \mu_1(t),$$

we obtain the differential equation

$$\mu_1'(t) + (\lambda_1 + \lambda_2)\mu_1(t) = \lambda_2. \tag{2.17}$$

The initial condition for this first order linear differential equation is

$$\mu_1(0) = 1, \tag{2.18}$$

which signifies that at time $t = 0$ the locus is in state 1. The solution of (2.17) under the initial condition (2.18) is easily seen to be[3]

$$\mu_1(t) = \frac{\lambda_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}. \tag{2.19}$$

[3] Expressions (2.19) and (2.20) are also derived in [2] by the use of various theorems in renewal theory. Our derivation uses first principles.

Thus,

$$\mu_2(t) = 1 - \mu_1(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2} - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}. \tag{2.20}$$

Since the lifetimes of all parts in a given state are equidistributed, we conclude that in the case of an infinite number of spares,

$$P\{\xi_1(t) = j, \xi_2(t) = |Q| - j\} = \binom{|Q|}{j} [\mu_1(t)]^j [\mu_2(t)]^{|Q|-j}. \tag{2.21}$$

For large values of t, (2.21) can be approximated by using the limiting relations

$$\lim_{t \to \infty} \mu_\nu(t) = \frac{\frac{1}{\lambda_\nu}}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}}, \qquad \nu = 1, 2.$$

These asymptotic formulae are not surprising, since the mean of the lifetime of a part in state ν is $1/\lambda_\nu$.

We consider now the case when $s < \infty$, which in turn falls naturally into the following two sub-cases:

(a) $k < |Q| - j$ in (2.5)

and

(b) $k = |Q| - j$ in (2.5).

Case (a) can occur only after stockout, while case (b) can occur either prior to or after stockout.

Letting $\eta(t)$ denote the number of complete failures up to time t in all of the loci combined, it is obvious that the time of stockout Y is given by

$$Y = \inf \{\tau : \eta(\tau) \geq s\}.$$

We wish to find the distribution function of Y. For $y \geq 0, F_s(y)$ denotes the probability that the time of occurrence of the sth complete failure will precede y. Equivalently, $1 - F_s(y)$ equals the probability that the number of complete failures occurring up to time y is less than s. Therefore,

$$1 - F_s(y) = P_0(y) + P_1(y) + \ldots + P_{s-1}(y), \tag{2.22}$$

where $P_\nu(y)$ denotes the probability that the total number of complete passages occurring in all the loci up to time y is equal to $\nu, \nu = 0, 1, \ldots, s - 1$.

In order to use formula (2.22) it is necessary to evaluate $P_\nu(y)$. To this end we first determine the density function, h, of the random variable $Z_1(O_j) + Z_2(O_j)$. Clearly,

$$h(t) = f * g(t),$$

where

$$f(t) = \begin{cases} \lambda_1 e^{-\lambda_1 t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

and

$$g(t) = \begin{cases} \lambda_2 e^{-\lambda_2 t}, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Thus,

$$h(t) = \int_0^t f(t-y)g(y)dy \tag{2.23}$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} [e^{-\lambda_2 t} - e^{-\lambda_1 t}], \qquad t \geq 0.$$

LEMMA 2.1: Let h^{*m} denote the m-fold convolution of h and set

$$H_m(t) = \int_0^t [1 - A(t-z)]h^{*m}(z)dz, \qquad m = 1, 2, \ldots, s, \tag{2.24}$$

where the function A is defined in (2.14). Then the probabilities $P_\nu(t)$, $\nu = 0, 1, \ldots, s$, satisfy the relation

$$P_\nu(t) = \sum_{\left\{k_j : \sum_{j=0} jk_j = \nu\right\}} \binom{|Q|}{k_0, k_1, \ldots, k_\nu} [1 - A(t)]^{k_0} [H_1(t)]^{k_1} [H_2(t)]^{k_2} \ldots [H_\nu(t)]^{k_\nu}. \tag{2.25}$$

PROOF: For ν complete failures in all of the loci combined to occur there must exist integers k_j, $0 \leq j \leq \nu$, such that there are precisely j failures in each of k_j loci, where

$$\sum_{j=1}^{\nu} jk_j = \nu,$$

and

$$k_0 = |Q| - \sum_{j=1}^{\nu} k_j.$$

We consider first the case where an infinite number of spares are available. Let $L_{0,r}$ denote the lifetime (i.e., time to a complete failure) of the part originally installed in locus λ_r and let $L_{m,r}$ denote the lifetime of the mth replacement in locus λ_r, $m=1, 2, 3, \ldots$. We set

$$\chi_{m,r}=L_{0,r}+L_{1,r}+\ldots+L_{m-1,r}$$

and

$$H_{m,r}(t)=P\{\chi_{m,r}<t,\ \chi_{m,r}+L_{m,r}\geq t\}.$$

It is clear that the density of $\chi_{m,r}$ is independent of r and is equal to the m-fold convolution of h. Moreover, $\chi_{m,r}$ and $L_{m,r}$ are independent random variables. Hence

$$H_{m,r}(t)=\int_0^t\int_{t-z}^{\infty} h^{*m}(z)h(w)\,dwdz$$

$$=\int_0^t [1-A(t-z)]h^{*m}(z)dz=H_m(t).$$

Clearly, $H_{m,r}(t)$ represents the probability of exactly m complete failures occurring in locus λ_r before time t.

If we denote by $M_\nu^{(\infty)}$ the event "exactly ν complete failures in all the loci combined prior to time t", in the infinite model (i.e., when there are an infinite number of spares), it follows that

$$P(M_\nu^{(\infty)})=\sum_{\left\{k_j:\ \sum_{j=0}^{\nu} jk_j=\nu\right\}}\binom{|Q|}{k_0,\ k_1,\ \ldots,\ k_\nu}[1-A(t)]^{k_0}[H_1(t)]^{k_1}[H_2(t)]^{k_2}\ \ldots\ [H_\nu(t)]^{k_\nu}=P_\nu(t).$$

In a similar fashion, it can be easily shown that the Lemma also holds for the finite model. Q.E.D.

We set

$$D(t)=P\{Z_2(O_j)\geq t\}$$

and

$$E(t)=P\{Z_2(O_j)<t\};$$

it is plain that the functions D and E do not depend on j. Clearly,

$$D(t)=\begin{cases}e^{-\lambda_2 t}, & t\geq 0\\ 0, & t<0\end{cases} \tag{2.26}$$

and

$$E(t)=\begin{cases}1-e^{-\lambda_2 t}, & t\geq 0\\ 0, & t<0.\end{cases} \tag{2.27}$$

We now consider case (a), $k < |Q| - j$ in (2.5). Let A_ν, $\nu = 1, 2, \ldots, |Q|$, denote the event that ν loci are in state 1 and $|Q| - \nu$ loci are in state a_2 at time of stockout, y. Plainly,

$$P(A_1 \cup A_2 \cup \ldots \cup A_{|Q|}) = \sum_{\nu=1}^{|Q|} P(A_\nu) = 1.$$

We note that ν cannot be zero, since at stockout we install the last working part and it is in state 1. Let B_m denote the event that the part which fails at time of stockout is in locus λ_m, $m = 1, 2, \ldots, |Q|$. It is clear from our assumptions that

$$P(B_m \cap B_n) = 0, \qquad \text{for } m \neq n,$$

that

$$\sum_{m=1}^{|Q|} P(B_m) = 1,$$

and by symmetry that

$$P(B_m) = \frac{1}{|Q|}, \qquad m = 1, 2, \ldots, |Q|.$$

Then

$$P(A_\nu) = \sum_{m=1}^{|Q|} P(A_\nu|B_m)P(B_m) = \frac{1}{|Q|} \sum_{m=1}^{|Q|} P(A_\nu|B_m).$$

It is plain that

$$P(A_\nu|B_m) = \binom{|Q| - 1}{\nu - 1} [\mu_1(y)]^{\nu-1}[\mu_2(y)]^{|Q|-\nu},$$

so that

$$P(A_\nu) = \binom{|Q| - 1}{\nu - 1} [\mu_1(y)]^{\nu-1}[\mu_2(y)]^{|Q|-\nu}, \qquad \nu = 1, 2, \ldots, |Q|. \tag{2.28}$$

We set

$$R_{j,k}(t) = \{\xi_1(t) = j, \qquad \xi_2(t) = k\}.$$

We first evaluate

$$P\{R_{j,k}(t)|Y = y\}, \qquad k < |Q| - j, \qquad t > y.$$

At time of stockout there exists an integer ν, $1 \leq \nu \leq |Q|$, such that ν loci are in state 1 and $|Q| - \nu$ loci are in state a_2. We denote the collection of ν loci which are in state 1 at time y by $\mathscr{A}$ and the collec-

tion of $|Q|-\nu$ loci which are in state a_2 at time y by $\mathscr{B}$. The occurrence of the event $R_{j,k}(t)$ specifies that at time $t>y$, j loci are in state 1 and k loci are in state a_2, and this can occur only in the following way: j loci from $\mathscr{A}$ remain in state 1, q loci from $\mathscr{A}$ fail partially, $\nu-j-q$ loci from $\mathscr{A}$ fail completely, $k-q$ loci from $\mathscr{B}$ remain in state a_2 and $|Q|-\nu-k+q$ loci from $\mathscr{B}$ fail to state 0, where $1\leq\nu\leq|Q|$, $j\leq\nu$, $0\leq q\leq$ min $(k, \nu-j)$ and $0\leq|Q|-\nu-k+q$. Recalling that the individual loci are independent of each other, we conclude that

$$(2.29)\quad P(R_{j,k}(t)|Y=y)=\sum_{\nu=1}^{|Q|}\sum_{q=0}^{k}\binom{\nu}{j,\,q,\,\nu-j-q}[C(t-y)]^{j}[B(t-y)]^{q}[A(t-y)]^{\nu-j-q}$$
$$\binom{|Q|-\nu}{k-q}[D(t-y)]^{k-q}[E(t-y)]^{|Q|-\nu-k+q}\binom{|Q|-1}{\nu-1}[\mu_1(y)]^{\nu-1}[\mu_2(y)]^{|Q|-\nu}.$$

We let $K_t(y)$ denote the right hand side of (2.29). Then, for $k<|Q|-j$

$$(2.30)\qquad P\{\xi_1(t)=j,\quad \xi_2(t)=k\}=\int_0^t K_t(y)\,dF_s(y).$$

We next consider case (b), $k=|Q|-j$ in (2.5). Plainly,

$$P\{\xi_1(t)=j,\quad \xi_2(t)=|Q|-j\}=P\{R_{j,|Q|-j}(t)|Y\geq t\}P\{Y\geq t\}+P\{R_{j,|Q|-j}(t)|Y<t\}P\{Y<t\}.$$

Prior to stockout our model is the probabilistic prototype of the case of infinite spares, and thus from (2.21) we conclude that

$$(2.31)\qquad P\{R_{j,|Q|-j}(t)|Y\geq t\}=\binom{|Q|}{j}[\mu_1(t)]^{j}[\mu_2(t)]^{|Q|-j}.$$

Since $k=|Q|-j$, we have $q=\nu-j$ and (2.29) becomes

$$(2.32)\quad P\{R_{j,|Q|-j}(t)|Y=y<t\}$$
$$=\sum_{\nu=1}^{|Q|}\binom{\nu}{j}[C(t-y)]^{j}[B(t-y)]^{\nu-j}[D(t-y)]^{|Q|-\nu}\binom{|Q|-1}{\nu-1}[\mu_1(y)]^{\nu-1}[\mu_2(y)]^{|Q|-\nu}.$$

We denote the right hand side of (2.32) by $L_t(y)$. Then

$$(2.33)\qquad P\{R_{j,|Q|-j}(t)|Y<t\}=\int_0^t L_t(y)\,dF_s(y).$$

Using (2.31) and (2.33), we conclude that

(2.34)

$$P\{\xi_1(t)=j,\xi_2(t)=|Q|-j\}=\binom{|Q|}{j}[\mu_1(t)]^{j}[\mu_2(t)]^{|Q|-j}[1-F_s(t)]+\left[\int_0^t L_t(y)\,dF_s(y)\right]F_s(t).$$

In conclusion, we note that the right-hand side of (2.2) and (2.4) can be evaluated using (2.13) for part types γ_i having $s_i = 0$ and (2.30) and (2.34) for part types with $s_i > 0$ and exponential lifetimes in both states.

One should note that all the calculations in this section can be generalized to the case of k levels of performance with exponential lives in each state.

REFERENCES

[1] Birnbaum, Z. W., J. D. Esary, and S. C. Saunders, "Multicomponent Systems and Structures and Their Reliability," Technometrics, *3*, 55–77 (1961).

[2] Cox, D. R., *Renewal Theory* (Methuen, London, 1962).

[3] Hirsch, W. M., M. Meisner, and C. Boll, "Cannibalization in Multicomponent Systems and the Theory of Reliability," Nav. Res. Log. Quart. *15*, 331–359 (1968).

[4] Simon, R. M., "Optimal Cannibalization Policies for Multicomponent Systems," SIAM J. Appl. Math *19*, 700–711 (1970).

A BAYESIAN APPROACH TO DEMAND ESTIMATION AND INVENTORY PROVISIONING*

George F. Brown, Jr. and Warren F. Rogers

Department of Management
Naval War College
Newport, Rhode Island

ABSTRACT

This article addresses the problem of explicitly taking into account uncertainty about the demand for spare parts in making inventory procurement and stockage decisions. The model described provides for a unified treatment of the closely related problems of statistical estimation of demand and resource allocation within the inventory system, and leads to an easily implemented, efficient method of determining requirements for spare parts both in the early provisioning phase and in later periods of operations when demand data have accumulated.

Analyses of the model's theoretical foundations and of sample outcomes of the model based upon data on parts intended for use in the F–14 lead to conclusions of great importance to both support planners and operations planners.

Finally, of particular significance is the ability afforded the planner by this model to quantify the impact on inventory system costs of varying levels of system reliability or management uncertainty as to projected system performance. This will provide an economic basis for analysis of such alternatives as early deployment, operational testing, and equipment redesign.

I. INTRODUCTION AND SUMMARY

Numerous studies† have demonstrated that the demand for aircraft spare parts is typically uncorrelated with identifiable program factors. In the absence of such deterministic predictors, statistical estimation procedures provide the best alternative means of estimating future requirements. Statistical estimation consists of specifying the probability distribution of demand which, in some sense, best explains the available data or, in the absence of data, best reflects the prior beliefs of the designer and the experience of the inventory manager. Having specified the probability distribution, it is necessary to determine the optimal inventory level as a function of the associated costs and budget constraints.

Typically the related problems of estimation and resource allocation are treated separately.** In simple inventory problems this is probably justified. However, when planning support of an extremely

*Work conducted under contract N00014–68–A–0091

†See, for example. Denicoff, M. and Haber, S., "A Study of Usage and Program Relationships for Aviation Repair Parts," The George Washington University, Logistics Research Project, Serial T–140/62 (7 Aug. 1962). The probability model we developed here is proposed in this reference and many others on empirical grounds. The fact that small correlations may be expected from data realized from a Poisson process does not appear to have been noted before.

**An exception is Zacks, S., "A Two-Echelon Multi-Station Inventory Model for Navy Applications," The George Washington University, Logistics Research Project, Technical Memorandum, Serial TM–15175 (31 July 1968), Zacks' approach is also Bayesian and uses the same probability model as that developed here.

complex weapons system, with very great numbers of parts of widely varying cost and uncertain performance, a unified treatment of these problems is essential. In short, the objective must be to specify the optimal inventory decision when system performance may be projected with only limited assurance.

This paper describes such a procedure for determining optimal inventory levels for aircraft spare parts. The procedure may be used before demand data have been generated by incorporating estimates developed at provisioning, and it provides for progressive updating of estimates as data become available. The model is simple to apply, extremely efficient, and requires only existing data sources. It is based on a few intuitive assumptions which have been repeatedly demonstrated to correspond closely to data on existing systems. The model may therefore be used with confidence, not only to determine inventory requirements, but perhaps more importantly to evaluate budgetary and operational implications of support policies.

An application of these procedures to a number of parts currently being provisioned for the F–14 is described. From this application and from theoretical consideration, a number of very important results are derived. The most important of these is that a spare parts inventory adequate to assure high system reliability early in system life will be very costly and extremely wasteful. The inventory required will be large but very little of it will actually be used. As data accumulate, however, it will be possible to design inventories to provide equal reliability assurance at greatly reduced cost. Thus, unless there are vital, overriding, operational requirements, the most desirable course of action is to accept low reliability in the early life of the system; this means procuring parts as needed until sufficient data have been accumulated to permit more economical inventory design.

Further significant results are summarized in the following paragraphs and are discussed in the remaining sections of the report.

1. The model described in this paper provides for a unified treatment of the closely related problems of statistical estimation of demand and resource allocation within the inventory system, which are typically treated separately. A frequent criticism of theoretical inventory models is that they do not reflect the uncertainty about the parameters which are inputs to the model—in particular, the probability distribution of demand. The procedure described here explicitly introduces such uncertainties into the inventory decision process.

2. Uncertainty about demand distributions can result from a number of factors. At the time of initial provisioning, estimates may be quite tentative due to the lack of any operational data on which to base them. Further, a system designed to operate worldwide, in a host of unpredictable environments, with a variety of maintenance procedures and skill levels supporting it, being employed in widely varying missions, can be expected to have not one, but many, rates of demand. Both forms of uncertainty are relevant to the inventory decision and are incorporated into the model described in this report. Furthermore, these two types of uncertainty imply different requirements for inventory support.

3. The model developed in this report enables the inventory manager to incorporate all of his particular knowledge about a deployment into the optimal decision process. Peculiarities about a particular deployment or a squadron's maintenance practices, as well as the size of the squadron and the projected flying hour program, can be reflected in the inputs to the model.

4. The effect of uncertainty about the demand rate is to increase the variance of the probability

distribution of demands. In turn, this high variance typically implies higher required levels of stockage, more frequent re-ordering, and, in general, higher costs of supporting the weapons system. This high variance and associated high support cost have been frequently reported in studies of Naval inventory systems. However, little guidance has been provided about what the Navy can do about these problems. Our model suggests a number of management procedures which can be employed to solve these problems beyond the usual suggestion that the equipment be redesigned so as to be made more reliable. In fact, we demonstrate, in some cases, that a reduction in uncertainty can be of more value than an equivalent increase in reliability. First, extensive operational testing can be undertaken to gather data which will lead to more certainty about demand rates. Planning to deploy extensively an untested weapons system and to support it for wartime usage will require high levels of inventory support. Furthermore, across parts, the higher the level of uncertainty, the greater will be the percentage of this inventory which will go unused. However, it is impossible, a priori, to tell exactly which parts will be used, so that extensive support across all such parts is required. Secondly, greater standardization of maintenance facilities and practices will reduce the variance in this demand and thus lead to lower inventory system costs. Finally, the ability of the inventory system manager to incorporate information peculiar to a particular squadron and deployment can reduce the variance in demand that the inventory system must protect against.

5. Numerous empirical studies of demand data have concluded that the observed pattern of demands over time correspond well with the realizations of a compound Poisson process. The explanations advanced to support this conjecture have largely been unsatisfying. The model developed in this paper, which follows from a few relatively mild assumptions, leads to one member of the compound Poisson family—the Negative Binomial distribution. Thus the results of this paper are supported by a wide body of previous empirical research.

6. A second major conclusion of previous empirical research has been that, with few exceptions, demands for spare parts are uncorrelated with program factors such as flight hours. The model developed in this report suggests that flight hours do enter into the determination of spare parts demands, but in a very complex and distinctly nonlinear way. We show that, in fact, the theoretical model developed here predicts the finding of a lack of correlation between flight hours and demand. The optimal inventory decisions generated in the model involve a highly complex interaction among the parameters of the demand distribution, relevant costs, and flight hours. Predictions of demand based upon simple linear relations between demands and flight hours are overly naive and are based upon a faulty premise.

Many of the mathematical results in this paper are well known. They are reproduced here both for completeness and because their implications for support policy are extremely important and have not been fully explored in the past.

The implementation of the procedures described in this paper should present little difficulty to managers of the Navy's inventory systems. All of the procedures employed in the analysis, including those for determining optimal inventory decisions and for incorporating new demand information as it becomes available, have been programmed and require only a few seconds of processing time. The decision rules have been shown to be of a particularly simple form and thus can be used by managers of deployed squadrons.

II. A MODEL OF SPARE PARTS DEMAND

The Probability Model

Inventory decisions in Navy Supply are typically based on point estimates of demand. When demands are subject to random variation, procedures based on point estimates will typically lead to poor decisions. An optimal inventory decision model must consider the full range of possible realizations of the random process which generates demands and their associated probabilities; the inventory model described in section IV does this.* In this section, we derive a probability model of demands which coincides well with empirical studies of demand data and is suitable for input in the inventory model.

Numerous empirical studies of demand data have been conducted.† Three conclusions emerge:

a. With very few exceptions, demands for spare parts are uncorrelated with program factors such as flying hours.
b. The Poisson distribution provides an adequate description of demands for parts exhibiting low demand rates.
c. The variance of demands for high usage parts over time is typically very large compared to their mean.

The latter observation has led to rejection of a simple Poisson model of the demand process for high usage rate parts since the Poisson distribution has identical mean and variance.

Several conjectures have been offered to explain this behavior and to justify the choice of one member of the compound Poisson family of distributions.** We have found these explanations unsatisfying either because they fail to correspond to operational experience or because the models they were advanced to support would be inappropriate if in fact the explanations were valid. Instead we show that one member of the compound Poisson family, the Negative Binomial distribution, follows logically from some rather mild assumptions and some practical constraints imposed by the nature of the estimation problem.

First we assume that demands for parts in non-overlapping time intervals are statistically independent. It is easily shown (cf. Feller [4]) that any distribution on the integers which satisfies this assumption is a member of the compound Poisson family.

Next we will assume that we may describe the uncertainty which exists about the anticipated rate of failures, λ, by assigning to it a probability distribution which summarizes designers', manufacturers', and support managers' best "guesses" as to the values of mean time between failures which may be realized when the equipment in question is placed in operation. The treatment of demand rate as a random variable may at first appear strange to those unacquainted with Bayesian methods. Justifica-

---

*A further treatment of the theoretical basis for this model is contained in Brown, George F., Jr., Corcoran, T. M., and Lloyd, R. M., "Inventory Models with Forecasting and Dependent Demand," *Management Science* (Mar. 1971), and "A Dynamic Inventory Model with Delivery Lag and Repair," Center for Naval Analyses, Professional Paper 3 (1969).

†For example, see Fawcett, W. M. and Gilbert, R. D., "Characteristics of Demand Distributions for Aircraft Spare Parts," General Dynamics Fort Worth Division Report ERR–FW–512 (Nov. 1966). Also see Youngs, J. W. T., Geisler, M. A., and Brown, B. B., "The Prediction of Demand for Aircraft Spare Parts Using the Method of Conditional Probabilities," RAND Corporation Report RM–1413 (Jan. 1955).

**For example, see Feeney, G. J. and Sherbrooke, C. C., "The $(s-1, S)$ Inventory Policy under Compound Poisson Demand," RAND Corporation Memorandum RM–4176–PR (Mar. 1966). The authors offer four conjectures to explain the high variability observed for recoverable item demand.

tion of this procedure is treated extensively in Raiffa and Schlaifer [5] and DeGroot [3]. In this particular application, however, it is intuitive that the underlying mean rate of failures which will be experienced when an equipment is employed in the Fleet should be expected to vary randomly with varying and unpredictable environments. We will refer to the distribution of λ as the prior distribution.

For any given realization of failure rate per unit of time, say λ, we will assume that the probability of observing more than one demand in any very small increment of time is itself vanishingly small.

With this last assumption and the assumption of independence between nonoverlapping time intervals we may conclude* that the conditional probability of observing k failures in any time increment t, *given that the rate* λ *holds* is given by:

$$P[k|\lambda] = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \cdot \tag{1}$$

If we denote our prior distribution on λ by $F(\lambda)$, then the unconditional probability of observing k failures in time t is:

$$P(k) = \int_0^\infty P(k|\lambda)\,dF(\lambda) \tag{2}$$

$$= \int_0^\infty \frac{(\lambda t)^k e^{-\lambda t}}{k!}\,dF(\lambda).$$

To this point, we have considered estimates of the distribution of demands based solely on prior considerations; that is, before demand data have been generated. Naturally, as demand data accumulate, we would wish to modify our prior beliefs about the mean demand rate to reflect this additional information. This is accomplished by an application of Bayes rule as follows. Let $f(\lambda) = \frac{dF(\lambda)}{d\lambda}$ be the prior density of λ and suppose that in each of n time periods, t_i, we have observed x_i demands, where $i = 1, 2, \ldots n$. Then the conditional density of λ, given the observations, is

$$f(\lambda|x_1, \ldots, x_n) = \frac{\prod_{i=1}^{n} P[x_i|\lambda t_i] \cdot f(\lambda)}{\int_0^\infty \prod_{i=1}^{n} P[x_i|y t_i] \cdot f(y)\,dy}. \tag{3}$$

We will refer to the conditional distribution of λ given the observations as the posterior distribution of λ.

With the additional information about demand rate summarized by the posterior distribution, the unconditional distribution of demands in Equation (2) now becomes

$$P[k] = \int_0^\infty \frac{(\lambda t)^k e^{-\lambda t}}{k!} f(\lambda|x_1, \ldots, x_n)\,d\lambda. \tag{4}$$

*For a rigorous statement of the postulates leading to this distribution, see Feller [4].

Choosing the Prior Distribution

To determine a suitable prior distribution $F(\lambda)$, Raiffa and Schlaifer [5] established the following desiderata:

"1. F should be analytically *tractable* in three respects: (a) it should be reasonably easy to determine the posterior distribution resulting from a given prior distribution and a given sample; (b) it should be possible to express, in convenient form, the expectations of some simple utility functions with respect to any member of F; (c) F should be closed in the sense that if the prior is a member of F, the posterior will also be a member of F.

2. F should be *rich*, so that there will exist a member of F capable of expressing the decision maker's prior information and beliefs.

3. F should be parametrizable in a manner which can be readily *interpreted*, so that it will be easy to verify that the chosen member of the family is really in close agreement with the decision maker's prior judgments about θ and not a mere artifact agreeing with one or two quantitative summarizations of these judgments."

It is of particular importance in this application that the criterion 1(c) apply. If we choose a prior distribution for which it did not, then the posterior distribution realized after each period of data collection would have an algebraic form differing from that of the preceding stage. Thus, extensive reprogramming would be required at each stage thereby effectively limiting the practical usefulness of the procedure. We therefore choose a family of distributions which satisfies 1(c) and examine its other properties.

A random variable λ is said to be distributed as the two parameter Gamma distribution with parameters α and β, denoted $G_{\alpha,\beta}$, if its density is

$$f(\lambda)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}. \tag{5}$$

If the parameter λ in the Poisson density given in Equation (5) has a prior distribution, $G_{\alpha,\beta}$, and if x_i, $i=1, 2, \ldots n$, are n independent samples from that Poisson process, then the posterior distribution of λ is again a Gamma distribution with revised parameters

$$\alpha'=\alpha+\sum_{i=1}^{n}x_i,\ \beta'=\beta+n.$$

Thus a Gamma prior satisfies criterion 1(c) and coincidentally 1(a).

For this application the utility function is defined implicitly by the inventory program and thus criterion 1(b) reduces to the requirement that the unconditional distribution of demands be computationally tractable. From Equations (2) and (5) we derive the unconditional distribution of demands as

$$P(k)=\int_0^{\infty}\frac{\lambda^k e^{-\lambda}}{k!}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}d\lambda \tag{6}$$

$$=\binom{\alpha+k-1}{k}\left(\frac{\beta}{\beta+1}\right)^{\alpha}\left(\frac{1}{\beta+1}\right)^{k},$$

the Negative Binomial distribution with parameters α and $\frac{\beta}{\beta+1}$. A simple recurrence relation which simplifies computation of the probabilities is given in section IV.

The Gamma family provides an extremely wide range of shapes, amply satisfying the second major criterion.

The final criterion deserves more extended consideration. The Gamma distribution is completely characterized by its mean and variance or by the mode and variance. The expected value (the mean) and the most likely value (the mode) of the rate of demands are probably meaningful concepts to an inventory manager or provisioner. It is doubtful, however, that variance is an equally meaningful concept and that prior estimates of it would really reflect their prior beliefs as to likely system performance. This question is treated in more detail in INS Study 37 [2].

It is of interest to note that while the distribution in Equation (6) is compound Poisson, the random process over the time parameter t is not. In fact, a nondegenerate mixture of Poisson processes cannot yield a compound Poisson process. The distribution in Equation (5) is, in fact, infinitely divisible in the parameter α not in t.[7]

III. SOME IMPORTANT IMPLICATIONS OF THE PROBABILITY MODEL

Deployment of New Weapons Systems

The model we have described has great intuitive appeal in that its development follows from a relatively few, mild assumptions, all of which appear consistent with operational experience. In addition there is strong (and plentiful) empirical evidence that the model accurately reflects real world experience. Predictions based on this model therefore merit serious consideration, particularly in view of their implications for wartime contingency planning.

The demand distribution given in Equation (7) has mean and variance

$$E(k) = \frac{\alpha t}{\beta},$$

$$\text{Var}\ (k) = \frac{\alpha t(\beta + t)}{\beta^2}.$$

The mean and variance of he prior distribution given in Equation (5) are

$$E(\lambda) = \frac{\alpha}{\beta},$$

$$\text{Var}\ (\lambda) = \frac{\alpha}{\beta^2}.$$

A large prior variance which implies a large uncertainty about λ is thus reflected in a large unconditional variance of demand. In addition the variance of the demand distribution increases quadratically

[7] We are grateful to Dr. Joseph Bram, who called this point to our attention.

in flying hours t. The immediate implication of increasing variance is that the probabilities of large demands also increase. To provide desired system reliability it is then necessary to procure larger inventories. But for fixed mean demand in the probability that no demands will in fact occur.* Thus the likelihood that expenditures will be wasted also becomes large. Of course across parts, it is impossible to tell with certainty which will be required and which will not.

Variance of demand is controlled by several factors. First there is the reliability of the system, reflected in the prior mean, $\frac{\alpha}{\beta}$. Then there is the variance of the prior, $\frac{\alpha}{\beta^2}$, which reflects the state of uncertainty about the current estimate of demand rate. Finally, there is the projected flying hour rate.

One conclusion is immediate. A new weapons system, incorporating "state of the art" equipment, whose performance may be projected only with great uncertainty, will require a large inventory of spare parts to ensure acceptable reliability. If, in addition, it is intended that the system be capable of sustaining an intensive wartime flying program, then the inventory must be expanded many times over. In fact, the sample calculations given in section IV indicate that even with the penalty cost fixed at the peacetime rate, which is no doubt unrealistically low, the war reserve inventory necessary to ensure high reliability in the absence of resupply would far exceed the levels normally maintained.

An inventory policy designed to provide for wartime employment early in the life of the system would not only be costly, but also extremely wasteful. It is important to realize the distinction between the planned inventory necessary to assure readiness and the usage which will actually be generated by the random process used in planning. The inventory must be designed to guard against demands whenever there is significant probability that they will occur. The demand actually realized will reflect the fact that there is also significant probability that a specific part will experience few or even zero demands.

The alternative is to defer some procurement decisions until the acquisition of demand data permits more reliable prediction of demand rates. As noted in section II, the posterior Gamma distribution of demand rate after n realizations of the process yielding demands X_i, $i = 1, 2, \ldots, n$, has parameters

$$\alpha + \sum_{i=1}^{n} X_i, \beta + n.$$

Then the posterior unconditional distribution of demands has variance

$$\text{Var } (k) = \frac{\left(\alpha + \sum_{i=1}^{n} X_i\right) t(\beta + n + t)}{(\beta + n)^2},$$

and thus the posterior variance decreases roughly as $\frac{1}{n}$. It follows that, in addition to allowing management to isolate those equipments whose realized reliability will dictate redesign action, deferral of

*The mean demand per flying hour is always less than one, so that for a fixed mean the increased mass at large values must be "balanced" by an increase at zero.

major commitment of resources enables us to design future inventories providing the desired level of readiness assurance, but at a greatly reduced cost.

Deferral of procurement, of course, implies acceptance of a reduced state of readiness in the early stages of the program so that enhanced readiness and the ability to respond to contingencies can be realized in later stages at acceptable cost. If, however, an initial high state of readiness and ability to respond to contingencies is deemed imperative, then the inventory should be planned realistically in the full realization that it will involve very great cost and potential waste.

Demand to Flying Hour Correlation

It has been noted earlier that estimates of the correlation between demand (or failures) and flying hours based on observed data are typically small. We now demonstrate that this should, in fact, be the expected outcome from data generated by our probability model.

Treating flying hours t as a random variable, we calculate the population correlation between t and k, the number of failures, as follows.

The covariance of t and k may be written

$$\begin{aligned}\text{Cov } (t, k) &= E(t(k - E(k))) \\ &= E(tk) - E(t)E(k).\end{aligned}$$

Then

$$\begin{aligned}E(k) &= E(E(k)|t) \\ &= E\left(\frac{\alpha}{\beta} t\right) \\ &= \frac{\alpha}{\beta} E(t),\end{aligned}$$

$$\begin{aligned}E(tk) &= E(E(tk|t)) \\ &= E\left(\frac{\alpha}{\beta} t\right) \\ &= \frac{\alpha}{\beta} E(t^2),\end{aligned}$$

whence

$$\text{Cov } (t, k) = \frac{\alpha}{\beta} \text{Var } (t).$$

Since

$$\text{Var } (k) = E(k^2) - E^2(k)$$

and

$$E(k^2) = E(Ek^2|t))$$

$$= E\left(\frac{\alpha t(\beta + t)}{\beta^2} + \left(\frac{\alpha}{\beta} t\right)^2\right)$$

$$= \frac{1}{\beta^2}\left(E(\alpha\beta t) + E(\alpha t^2) + \alpha^2 E(t^2)\right),$$

$$\text{Var } (k) = \frac{\alpha}{\beta} E(t) + \frac{\alpha}{\beta^2} E(t)^2 + \frac{\alpha^2}{\beta^2} E(t^2) - \frac{\alpha^2}{\beta^2} E^2(t)$$

$$= \frac{\alpha}{\beta}\left(E(t) + \frac{1}{\beta} E(t^2) + \frac{\alpha}{\beta} \text{Var } (t)\right).$$

Thus

$$\text{Corr } (t, k) = \left(\frac{\frac{\alpha}{\beta} \text{Var } (t)}{E(t) + \frac{1}{\beta} E(t^2) + \frac{\alpha}{\beta} \text{Var } (t)}\right)^{1/2}$$

$$= \left(\frac{1}{1 + \frac{\beta}{\alpha} \frac{E(t)}{\text{Var } (t)} + \frac{1}{\alpha} \frac{E(t^2)}{\text{Var } (t)}}\right)^{1/2}$$

$$< \left(\frac{1}{1 + \frac{\beta}{\alpha} \frac{E(t)}{\text{Var } (t)}}\right)^{1/2}.$$

Now $\frac{\alpha}{\beta}$ is the expected number of demands per flying hour, which is typically very small, so that strong correlation will exist only if the variance of t is large relative to its mean. We are thus led to the somewhat vacuous conclusion that correlations will be large only if flying hours are extremely variable and thus cannot be predicted with assurance.

From our earlier discussion of the probability model, it should be apparent that demands are not statistically independent of flying hours, but it should also be clear that the dependence is distinctly nonlinear. The optimal inventory decisions generated in the model involve highly complex interactions among the parameters of the distribution, the relevant costs, and flying hours. Predictions of demand based on simple linear relations between demands and flying hours are overly naive and, as the discussion here shows, are based on a faulty premise.

IV. AN APPLICATION TO INVENTORY MANAGEMENT

Calculation of Probabilities Required for the Inventory Model

The inventory model employed in this analysis employs dynamic programming techniques to determine the optimal order size in each period, y_t, and the optimal initial stockage, I_0, using a single

state variable, J_t, the number of items on hand, on order (but not yet delivered), and in repair at the end of period t. Defining $f_t(J)$ as the total discounted expected costs under control of the inventory manager from period t to the end of the planning horizon, given J units on hand, on order, and in repair, following an optimal policy, the following recursive relationship may be used to determine y_t and I_0:

$$f_t(J_{t-1}) = \begin{cases} 0 & \text{for } t = T-l_1+1, \ldots, T+1 \\ \min\limits_{y_t \geq 0} \{K\delta(y_t) + \alpha E f_{t+1}[J_{t-1}+y_t-D_t+R_t] & \\ \qquad + \alpha^{l_1} G_{t+l_1}(J_{t-1}+y_t)\} & \text{for } t = 1, \ldots, T-l_1 \\ \alpha f_1(I_0) + G'(I_0) + K\delta(I_0) & \text{for } t = 0, \end{cases}$$

where

$G_{t+l_1}(J)$ = expected holding and penalty costs during period $t+l_1$, given J units on hand, on order, and in repair at the beginning of period t;

$G'(I_0)$ = expected holding and penalty costs during periods 1, 2, . . ., l_1, plus initial holding costs, given a starting on hand inventory of I_0.

This inventory model is designed to be used with any distribution of demand. Three probability calculations are required:

1. Probability of k_1 failures in n decision periods.
2. Probability of k_2 nonrepairable failures in m decision periods, given a probability p that a failed part is repairable.
3. The probability that in two nonoverlapping time intervals, t_1 and t_2 of length n and m periods, respectively, a total of k failures and nonrepairable failures will be observed where all failures are recorded in t_1 and only nonrepairable failures are recorded in t_2.

The first calculation follows immediately from Equation (6).

If t is the number of flight hours per aircraft per period and the distribution of λ, the rate of failures, is $G_{\alpha,\beta}$, then

$$(7) \qquad P[k \text{ failures in one aircraft in one period}] = \int_0^\infty \frac{(\lambda t)^k e^{-\lambda t}}{k!} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda$$

$$= \binom{\alpha+k-1}{k} \left(\frac{\beta}{\beta+t}\right)^\alpha \left(\frac{t}{\beta+t}\right)^k.$$

Then if r is the number of aircraft,

$$(8) \qquad P[k_1 \text{ failures in } n \text{ periods}]^{\text{def}} = P_{n,r}(k_1)$$

$$= \binom{nr\alpha+k_1-1}{k_1} \left(\frac{\beta}{\beta+t}\right)^{nr\alpha} \left(\frac{t}{\beta+t}\right)^{k_1}.$$

This result follows because the Negative Binomial is reproductive with respect to α. Note that we are modeling the n period, r aircraft process as the sum of $n \cdot r$ independent replications of the basic process

in equation (7). An alternative formulation would result if we considered a single process and *nrt* flying hours as follows:

$$P_{nr}(k) = \int_0^\infty \frac{(\lambda nrt)^k e^{-\lambda nrt}}{k!}\, dF(\lambda) \tag{9}$$

$$= \binom{\alpha+k-1}{k} \left(\frac{\beta}{\beta+nrt}\right)^\alpha \left(\frac{nrt}{\beta+nrt}\right)^k.$$

However, the intent here is to incorporate the uncertainty about the value of λ which arises in large part from the variability and unpredictability of the environments in which individual aircraft will operate at different times, and thus the representation in equation (8) is appropriate.

For the second calculation we require the following.

THEOREM. If failures are distributed as the Negative Binomial with parameters α and $\left(\frac{\beta}{\beta+t}\right)$ and the probability that a failed part is repairable is p, then the distribution of nonrepairable failures is again Negative Binomial with parameters α and $\left(\frac{\beta}{\beta+(1-p)t}\right)$.

PROOF:

$$P[k \text{ nonrepairable failures}] = \sum_{x=k}^{\infty} P[k \text{ nonrepairable failures} \mid x \text{ failures}]$$

$$\cdot \binom{\alpha+x-1}{x} \left(\frac{\beta}{\beta+t}\right)^\alpha \left(\frac{t}{\beta+t}\right)^x$$

$$= \sum_{x=k}^{\infty} \binom{x}{k} p^{x-k}(1-p)^k \binom{\alpha+x-1}{x} \cdot \left(\frac{\beta}{\beta+t}\right)^\alpha \left(\frac{t}{\beta+t}\right)^x$$

$$= \left(\frac{(1-p)t}{\beta+t}\right)^k \left(\frac{\beta}{\beta+t}\right)^\alpha \frac{1}{k!} \sum_{x=0}^{\infty} \frac{(\alpha+x+k-1)!}{x!(\alpha-1)!} \cdot \left(\frac{pt}{\beta+t}\right)^x$$

$$= \left(\frac{(1-p)t}{\beta+t}\right)^k \left(\frac{\beta}{\beta+t}\right)^\alpha \binom{\alpha+k-1}{k} \cdot \sum_{x=0}^{\infty} \binom{\alpha+x+k-1}{x} \left(\frac{pt}{\beta+t}\right)^x$$

$$= \left(\frac{(1-p)t}{\beta+t}\right)^k \left(\frac{\beta}{\beta+t}\right)^\alpha \binom{\alpha+k-1}{k} \left(1-\frac{pt}{\beta+t}\right)^{-(\alpha+k)}$$

$$= \binom{\alpha+k-1}{k} \left(\frac{\beta}{\beta+(1-p)t}\right)^\alpha \left(\frac{(1-p)t}{\beta+(1-p)t}\right)^k. \qquad \text{Q.E.D.}$$

Again exploiting the reproductive quality of the Negative Binomial, we have the probability that r aircraft generate k_2 nonrepairable failures in m periods:

$$P_{m,r}(k_2) = \binom{mr\alpha + k_2 - 1}{k_2} \left(\frac{\beta}{\beta + (1-p)t}\right)^{mr\alpha} \left(\frac{(1-p)t}{\beta + (1-p)t}\right)^{k_2}. \tag{10}$$

The third calculation may now be carried out directly:

$$P[k_1 + k_2 = k] = \sum_{j=0}^{k} P_{n,r}(k-j) P_{m,r}(j). \tag{11}$$

Calculations are simplified by use of the following simple recurrence. If K is distributed as the Negative Binomial with parameters a and b,

$$P[K = k+1] = \frac{a+k}{k+1} (1-b) P[K = k]$$

$$P[K = 0] = b^a.$$

Empirical Results for F-14 Parts

This section contains empirical results from an application of the procedures described in the preceding section to parts currently being provisioned for the F−14. While a number of the results summarized here have been predicted by the theory, they give illustrations of the great magnitude of the effects of these influences.

Table I presents results for the F−14 nose landing gear as a function of the degree of uncertainty about the failure rate.* A wide range of the parameters α and β was chosen to illustrate prior distributions all having the same mean, but with increasing uncertainty (or variance). Each of these prior distributions implies the parameters of the unconditional demand distribution (the Negative Binomial), which are also tabulated. Finally, three outputs of the inventory model are included:

(1) The optimal initial stockage:
(2) The optimal re-order policy;

which is of the (s, S) form. If X is the stock on hand, on order, and in repair at the beginning of a period, the optimal reorder policy is do not order if $X \geq s$ and order $S - X$ if $X < s$.

(3) The expected inventory system costs,

over a 6-month cruise, if an optimal policy is followed (for a deckload of 24 aircraft, each flying an average of 1 hour per day).

*Similar tables for additional parts appear in INS Study 37.

TABLE I. *Optimal Inventory Policies for the Nose Landing Gear*

α	14	0.224	0.112	0.056	0.028	0.021	0.014	0.0105	0.007	0.0035
β	1000	16	8	4	2	1.5	1	0.75	0.5	0.25
Mean of prior distribution	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
Variance of prior distribution	0.014×10^{-3}	0.000875	0.00175	0.0035	0.007	0.0093	0.014	0.0187	0.028	0.056
Mean of unconditional demand distribution	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
Variance of unconditional demand distribution	0.014	0.014875	0.01575	0.017	0.021	0.023	0.028	0.044	0.056	0.07
Optimal initial stockage	2	2	2	2	3	3	3	3	4	6
Optimal (s, S) reorder policy	(1, 2)	(1, 2)	(1, 2)	(1, 2)	(2, 3)	(2, 3)	(2, 3)	(2, 3)	(3, 4)	(5, 6)
Expected inventory system costs ($)	79,476	82,027	84,822	90,298	98,278	103,201	112,566	121,114	134,767	153,968

The first column in the table corresponds closely with a simple Poisson distribution. The mean and the variance of the unconditional demand distribution are virtually identical. This results from the fact that the variance of the prior is extremely small; failure rates different from the prior mean are felt to be very unlikely and are given little weight. Moving across each table, these results are presented for cases in which the uncertainty about the true failure rate grows larger; thus the variance in the unconditional demand distribution also increases. As the uncertainty grows, the inventory system costs and the required stockage levels also increase rapidly. These empirical results clearly demonstrate the high costs associated with uncertainty about the demand distribution, and show the importance of the management actions which can be taken to reduce this uncertainty. Early in the provisioning process, it is unlikely that there would be great confidence about the demand rate; thus, if parts are procured at this time, the high inventory system costs associated with uncertainty must be incurred. Planning for full deployment of a weapons system before much information about it is gathered could potentially require support at a cost much higher than would be required later in its service life.

The great costs associated with uncertainty are further illustrated in Table II. There, inventory system costs are presented for a range of means and variances of the unconditional demand distribution. A decrease in the mean represents an increase in "reliability," while an increase in the variance represents a greater degree of uncertainty about the mean. The surprising conclusion that comes from this table is that uncertainty may be more expensive to the Navy than unreliability. Changes in the mean (holding variance constant) affect inventory system costs very little, while changes in the variance (holding the mean constant) produce much greater cost increases. Hence programs to redesign equipment may have very little impact unless greater certainty results from the redesign process.

TABLE II. *Effects of Reliability and Uncertainty on Expected Inventory System Costs**

Variance \ Mean	0.010	0.012	0.014	0.016	0.018
0.010	64,716				
0.012	71,418	72,003			
0.014	77,812	78,305	79,439		
0.016	84,010	84,595	85,604	87,142	
0.018	89,919	90,755	91,861	93,316	95,196

*For nose landing gear.

Table III shows the effect of changes in the flying hour program on optimal stockage and reorder policies, and on expected inventory system costs. We have previously shown that, while demands cannot be predicated by means of a naive relationship with the flying hour program, the flying hour program does enter in the demand distribution in a complex way, and thus must affect resource allocation decisions. These points are clearly demonstrated in the table—higher flying hour programs require greater inventory investment and are much more expensive. Furthermore, the greater the uncertainty about the system, the greater will be the increase in this investment. Wartime flying hour programs with a system

TABLE III. *Effects of Changes in Flight Hour Program**

Flight hours**	Optimal initial stockage	Optimal reorder policy	Expected inventory system costs
0.5	2	(1, 2)	57,685
0.75	2	(1, 2)	73,000
1.0	2	(1, 2)	90,298
1.25	3	(2, 3)	105,454
1.5	3	(2, 3)	120,837
2.0	4	(3, 4)	151,575
2.5	5	(4, 5)	180,982
3.0	6	(5, 6)	209,899
4.0	7	(7, 8)	267,278
5.0	8	(8, 9)	321,050

*For nose landing gear, $\alpha = 0.056$, $\beta = 4.0$.
**Average daily flight hours per aircraft.

about which there is great uncertainty will require enormous inventory investments. The potential value of management actions aimed at reducing uncertainty again becomes apparent.

Finally, Table IV illustrates the fact that greater uncertainty about a system also leads to greater potential wastage. Presented in the table are the probabilities of demands of various sizes on a single day for a system with mean of 0.014 and the variances listed. As the variance increases, two things happen: the probability of zero demands increases and the probability of large demands increases. Thus, while the inventory decision must provide insurance against these high demands and the associated lessening of readiness, the probability of this insurance being wasted also increases. While the changes in the probability of zero demands seem small numerically, over an extended period of time, these small changes become significant. Again, a reduction in uncertainty will lead to a decrease in both required stockage and potential wastage.

TABLE IV. *The Negative Binomial Density Function*

Variance	Probability that demands* on a single day will be:												
	0	1	2	3	4	5	6	7	8	9	10	11	≥12
0.014	0.98610	0.01379	0.00010	0.00000									
0.014875	0.98651	0.01300	0.00047	0.00002	0.00000								
0.01575	0.98689	0.01228	0.00076	0.00006	0.00001	0.00000							
0.17	0.98758	0.01106	0.00117	0.00016	0.00002	0.00000							
0.021	0.98871	0.00923	0.00158	0.00036	0.00009	0.00002	0.00001	0.00000					
0.023	0.98933	0.00831	0.00170	0.00046	0.00014	0.00004	0.00001	0.00001	0.00000				
0.028	0.99034	0.00693	0.00176	0.00059	0.00022	0.00009	0.00004	0.00002	0.00001	0.00000			
0.044	0.99114	0.00595	0.00172	0.00066	0.00028	0.00013	0.00006	0.00003	0.00002	0.00001	0.00000		
0.056	0.99234	0.00463	0.00155	0.00069	0.00035	0.00019	0.00010	0.00006	0.00003	0.00002	0.00001	0.00001	0.00000
0.070	0.99438	0.00278	0.00112	0.00060	0.00036	0.00023	0.00015	0.00011	0.00007	0.00005	0.00004	0.00003	0.00006

*For part with mean 0.014 (such as nose landing gear).

REFERENCES

[1] Brown, G. F., Jr. and Perlman, B. L., "Optimal Inventory Management for Naval Aviation Support," Center for Naval Analyses, Research Contribution 186 (1971).

[2] Center for Naval Analyses, "F−14/Phoenix Provisioning: A Study of Initial Spare Part Provisioning of A New Weapons System," Institute of Naval Studies Study 37 (1972).

[3] DeGroot, M. H., *Optimal Statistical Decisions* (McGraw-Hill Book Co., New York, 1970).

[4] Feller, W., *An Introduction to Probability Theory and Its Applications* (John Wiley and Sons, Inc., New York), Vol. I, 1957, Vol. II, 1966.

[5] Raiffa, H. and Schlaifer, R., *Applied Statistical Decision Theory* (the M.I.T. Press, Cambridge, Mass., 1961).

READINESS AND THE OPTIMAL REDEPLOYMENT OF RESOURCES*

Seymour Kaplan

Department of Industrial Engineering and Operations Research
New York University

ABSTRACT

This paper considers the problem of the optimal redeployment of a resource among different geographical locations. Initially, it is assumed that at each location i, $i=1, \ldots, n$, the level of availability of the resource is given by $a_i \geq 0$. At time $t>0$, requirements $R_j(t) \geq 0$ are imposed on each location which, in general, will differ from the a_i. The resource can be transported from any one location to any other in magnitudes which will depend on t and the distance between these locations. It is assumed that $\Sigma R_j > \Sigma a_i$.

The objective function considers, in addition to transportation costs incurred by reallocation, the degree to which the resource availabilities after redeployment differ from the requirements. We shall associate the unavailabilities at the locations with the unreadiness of the system and discuss the optimal redeployment in terms of the minimization of the following functional forms: $\sum_{j=1}^{n} k_j(R_j-y_j)$ + transportation costs, $\max_j [k_j(R_j-y_j)]$ + transportation costs, and $\sum_{j=1}^{n} k_j(R_j-y_j)^2$ + transportation costs. The variables y_j represent the final amount of the resource available at location j. No benefits are assumed to accrue at any location if $y_j > R_j$. A numerical three location example is given and solved for the linear objective.

PROBLEM

Suppose there are n geographical locations where an organization requires varying levels of a resource (manpower, fuel, equipment). The requirements for this resource are assumed to change as sudden demands for the resource brought about by changing economic, political, or natural conditions are created. For example, natural disasters such as floods may create a need for certain types of rescue equipment at various flood locations. To satisfy the needs at any one location, the resource may be obtained locally or from any other locations where availability exists. There are limitations on the magnitudes of the resource which may be transported from location i to location j. These limitations depend on the allowable time, t, for reallocation to take place as well as the distance between locations. In the present problem, t is fixed and given so that the limitations are given constants.

We shall consider several types of objective functions (to be discussed below) which we wish to associate partially with the degree of unreadiness of the system. That is, we consider several different measures of unreadiness and investigate how the optimal reallocation changes with these measures. In addition to the costs incurred as a result of unreadiness, we assume that the physical process of reallocation also results in transportation costs. The weighted sum of these two types of costs will

*Prepared under Contract N00014-67-A-0467-0028 for the Office of Naval Research.

constitute the objective function. In each case, it is assumed that ending up with more of a resource than required at a location does not result in any benefits. Also the problem is deterministic and contains no stochastic elements.

DEFINITIONS

Let

x_{ij} = the amount of the resource to be transported from location i to location j;
y_j = the final level of the resource at location j;
c_{ij} = the cost of transporting one unit of the resource from location i to location j, $c_{ij} \geq 0$;
a_i = the initial availability of the resource at location i, $a_i \geq 0$;
$R_j(t)$ = the requirement of the resource by time t at location j, $R_j(t) = R_j \geq 0$, where t is assumed fixed;
$M_{ij}(t)$ = the maximum allowable magnitude of the resource that can be shipped from i to j in an interval of length t ($M_{ij}(t) = M_{ij} \geq 0$);
k_j = the relative importance of location j insofar as resource insufficiency at that location is concerned. The greater k_j, the more critical an insufficiency at location j, $k_j \geq 0$.

It is assumed that

$$\sum_{j=1}^{n} R_j > \sum_{i=1}^{n} a_i.$$

PROBLEM FORMULATION

The problem to be solved can be set up in a transportation type format where each location is considered as both an origin and destination. The constraints state that the amount of product to be sent from location i cannot exceed a_i, the amount received by any location is equal to y_j, where y_j cannot exceed R_j and the amount shipped from any location to any other is limited by the M_{ij}. Thus, we obtain:

$$\text{(1)} \qquad \text{Min } z = f(R_i, y_j) + \sum_i \sum_j c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} \leq a_i; \qquad i = 1 \ . \ . \ . \ n$$

$$\sum_{i=1}^{n} x_{ij} = y_j; \qquad j = 1, \ . \ . \ . \ n$$

$$y_j \leq R_j \qquad j = 1 \ . \ . \ . \ n$$

$$x_{ij} \leq M_{ij} \qquad \text{all } i, j$$

$$x_{ij} \geq 0, \quad \text{all } i, j; \ y_j \geq 0 j = 1, \ . \ . \ . \ n.$$

The objective will be referred to as the unreadiness function and we shall consider and discuss several different mathematical forms of this function. Note that the y_j are problem variables. If we take a linear objective function of the form

$$z=\sum_{j=1}^{n} k_j(R_j-y_j)+\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij},$$

it will be seen that the problem can be reduced to a standard capacitated transportation problem. Let $u_j=R_j-y_j$. Then (1) becomes:

(2)
$$\text{Min } z=\sum_{j=1}^{n} k_j u_j+\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} \leqslant a_i \qquad i=1, \ldots n.$$

$$\sum_{i=1}^{n} x_{ij}+u_j=R_j \qquad j=1, \ldots n$$

$$x_{ij} \leqslant M_{ij}$$

$$x_{ij} \geqslant 0, \qquad u_j \geqslant 0.$$

If the u_j are considered as the amounts shipped from an additional fictitious origin then the problem can be considered as one where the unreadiness costs (the k_j) are associated with shipping from the additional origin. If the availability at this origin is considered to be a_{n+1}, where a_{n+1} may be set equal to some large value $\left(\sum_{j=1}^{n} R_j \text{ will do}\right)$, then an additional origin constraint of the form

$$\sum_{j=1}^{n} u_j \leqslant a_{n+1}$$

puts the problem into a format with $n+1$ origin constraints and n destination constraints. The problem may be interpreted insofar as unreadiness is concerned, as one where we wish to avoid shipping from origin $(n+1)$ as much as possible. If the $\sum_{i=1}^{n} x_{ij}=R_j$ then the requirement at j can be met without unreadiness penalty. If $\sum_{i=1}^{n} x_{ij}<R_j$, then a penalty due to unreadiness is incurred at location j. Or, one may state the problem as one where unreadiness costs are only associated with slack variables in the destination constraints when the problem is cast in the form:

(3)
$$\text{Min} \sum_{i=1}^{n+1}\sum_{j=1}^{n} c_{ij}x_{ij},$$

subject to

$$\sum_{j=1}^{n} x_{ij} \leqslant a_i,\ i=1,\ \ldots\ n+1.$$

$$\sum_{i=1}^{n+1} x_{ij} \leqslant R_j,\ j=1,\ \ldots\ n.$$

$$x_{ij} \leqslant M_{ij}$$

$$x_{ij} \geqslant 0,\ i=1,\ \ldots\ n+1; \qquad j=1,\ \ldots\ n,$$

and where

$$c_{n+1,j}=k_j.$$

To finally state the problem in the standard transportation format, consider an additional fictitious destination such that the slack variables of the origin constraints represent the amounts of the resource shipped to this destination. Call the slack variables $x_{i,\,n+1}$, where $i=1,\ \ldots\ n+1$. Then the problem becomes

$$\text{Min} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} c_{ij}x_{ij},$$

subject to

$$\sum_{j=1}^{n+1} x_{ij}=a_i, \qquad i=1, 2,\ \ldots\ n+1$$

$$\sum_{i=1}^{n+1} x_{ij}=R_j, \qquad j=1, 2,\ \ldots\ n+1$$

$$0 \leqslant x_{ij} \leqslant M_{ij} \qquad \text{all } i, j.$$

In this problem,

$$a_{n+1}=\sum_{j=1}^{n} R_j$$

$$R_{n+1}=\sum_{i=1}^{n+1} a_i=\sum_{j=1}^{n} R_j=\sum_{i=1}^{n} a_i \left(\text{ so that } \sum_{i=1}^{n+1} a_i=\sum_{j=1}^{n+1} R_j\right).$$

Also the $M_{jj}=\text{Min}\ (a_j, R_j)$ so that if $R_j < a_j$ location j will only end up with R_j, whereas if $R_j \geqslant a_j$, the entire availability can remain. Since the x_{ii} represent shipping from a location to itself, we shall assume that $c_{ii}=0$. Also, we take $M_{n+1,\,j}=R_j$ so that if necessary, up to R_j units will be sent to destination j from origin $n+1$; and $M_{i,\,n+1}=a_i$. Finally $c_{i,\,n+1}=0$, $i=1,\ \ldots\ n+1$.

Assuming that a feasible solution exists, the above problem can be solved as a capacitated transportation problem with $n+1$ origins and $n+1$ destinations.

When the objective is in the form

$$z = \operatorname{Max}_j \left[k_j(R_j - y_j)\right] + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij},$$

we can convert the problem to a linear program, but not a transportation problem by noting that

$$z = \operatorname{Max}_j \left[k_j(R_j - y_j) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \right]$$

After making the transformation $u_j = R_j - y_j$ as before, the problem is equivalent to the following linear program:

(4) $$\text{Min } v$$

$$k_j u_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \leqslant v, \qquad j = 1, 2, \ldots n + \text{the other constraints of (2).}$$

The above objective is often referred to as a minimax objective and can occur in curve fitting and regression problems as well as in the present context. See [4] for example.

With a quadratic objective of the form

$$z = \sum_{j=1}^{n} k_j(R_j - y_j)^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij},$$

the problem may be solved as a quadratic program after letting $u_j = R_j - y_j$, since the quadratic form

$$\sum_{j=1}^{n} k_j u_j^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$$

is positive definite (k_j, $c_{ij} \geqslant 0$ and the form cannot have the value zero since $a_j < R_j$). Wolfe's method for quadratic programming is a convenient procedure to use. See [43].

It should be noted that with the minimax objective and the quadratic, the problem can be solved via simplex tableaus. The minimax problem requires n additional constraints above the $n+1$ origin and $n+1$ destination constraints, where $n =$ the number of locations. The quadratic problem, via the Wolfe technique requires $(n+1)^2$ additional constraints, corresponding to the number of variables in the problem with $n+1$ origins and $n+1$ destinations.

OBJECTIVE FUNCTION

The objective function is one which transforms the cost of unreadiness into costs associated with transportation and assumes such a cost is additive to the transportation costs. The great difficulty of such a procedure is of course in developing meaningful empirical procedures for such a transformation.

If we consider that the objective functions represent a disutility to the organization then we are assuming that the disutility due to unreadiness is additive to that of transportation cost. We are here essentially dealing with the problem of decision making with respect to multiple objectives and encounter the usual difficulties when doing so. See [1] for examples.

In the context of the present problem, we consider the disutility due to unreadiness to be the major concern and include the transportation costs because the formulation is more general, no difficulties are added to the problem in solution, and because such costs may, in fact, influence the optimal reallocation if some of them are sufficiently large. However, the problem can also be considered with all $c_{ij}=0$ so that the unreadiness disutility is the only consideration.

The linear objective function for unreadiness assumes that the overall unreadiness is measured as a weighted sum of the insufficiencies in the supply of the resource, the weight taken over the different geographical locations. The weights may be normalized and could be estimated by a variety of techniques relating to the problem of decision making with respect to multiple criteria. In essence, we are assuming that the organization has an additive linear disutility function with respect to resource insufficiencies.

With the objective function which minimizes the maximum insufficiency, the measure of unreadiness is related to the worst possible insufficiency and is essentially a "conservative" criterion. For any optimum solution to this problem, the average insufficiency taken over locations will, in general, be expected to be greater than with the previous criterion.

With the quadratic unreadiness objective, the measure of course penalizes locations more severely for insufficiencies > 1 than does the linear function. Here again the assumption is of an additive utility function taken over locations.

Much of which type of objective, of the three discussed, as well as others, will of course depend on the nature of the resource and how it is combined or used with other resources. Resources such as aircraft fuel may, in short supply, penalize short run operations much more severely than resources such as certain food items. In the latter case the min-max objective might be more appropriate since we might be interested in the shortage of such resources not getting out of "control" anywhere and trying to keep the worst possible shortage as low as possible.

EXTENSION TO MULTIPLE RESOURCES

If we assume that a simultaneous shortage of two or more resources affects the ability of the organization to carry out its mission to an extent greater than or equal to that of one resource, then we can postulate a variety of models for describing this simultaneous shortage.

Much will depend on how the resources interact with each other in carrying out functions. Thus, certain levels of pilot and airplane shortages simultaneously may not affect the readiness much more than the given shortage level of just one of these, whereas corresponding shortages of pilots and ASW equipment may affect the readiness of a unit in an additive manner.

An additive situation would seem appropriate when the resources in question were used for what may be termed "independent" missions, where the resources needed for one mission are unrelated to those needed for the others. Of course, in a real sense, no two missions of an organization during a particular period of time are truly independent. However, if the additive model seems appropriate, the problem could be handled by including another summation in the objective function over resources

and adding additional constraints for each resource. Thus, the form of the objective function for the linear unreadiness model would be:

$$\text{Min } z = \sum_{l=1}^{q} \sum_{j=1}^{n} k_{lj}(R_{lj} - y_{lj}) + \sum_{l=1}^{q} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ijl} X_{ijl},$$

where there are q resources, and where the subscript l refers to the lth resource.

Nonadditive situations would involve certain nonlinearities in formulation and are beyond the scope of this paper.

EXAMPLE

We shall illustrate the solution for the linear objective function with an example. Consider the following reallocation problem with three locations, set up in a tableau format as follows:

Location	1	2	3	a_i
1	0 / 4	0.01 / 2	0.02 / 2	4
2	0.02 / 3	0 / 6	0.02 / 3	6
3	0.02 / 1	0.01 / 1	0 / 7	7
R_j	6	8	8	$\Sigma a_i = 17$
k_j	0.4	0.3	0.2	$\Sigma R_j = 22$

The numbers in the upper left of each cell of the 3×3 location matrix indicate the transportation costs, while those in the lower right indicate the capacity of each route, i.e., $c_{12} = 0.01$, $x_{12} \leq 2$. The overall requirement is for 22 units, whereas the overall availability is 17.

We shall solve the problem by means of the primal-dual method for the capacitated transportation problem and the notation and tableau format of Hadley [2].

The problem requires six tableaus for solution; they are shown in the appendix. The optimal minimum cost solution is found by transporting one unit from location three to location one and one unit from location three to location two. The optimal redeployment can be read off the final tableau (Table 1) reproduced below. The values in the circles of the fourth row cells (0_4) corresponding to the fictitious origin, show the final deficiencies at each location, i.e., $R_1 - y_1 = 1$, $R_2 - y_2 = 1$, $R_3 - y_3 = 3$. (The 17 is the excess going to the fictitious destination). The values in the circles on the off-diagonal elements indicate the redeployments. In this problem the value of the objective function is $z_{\min} = 1.33$.

TABLE 1. *Final Tableau for the Example.*

For notation, see pp. 358 and 397 of [2].
Solution $x_{31}=1$, $x_{32}=1$, $z=1.33$.

		D_1		D_2		D_3		D_4				
	v_j / u_i	0.00		0.00		−0.20		−0.40		a_i / x_s^i	δ_i	γ_i
0_1	0.00	0.00	4	0.01	2	0.02	2	0.00	4	4		
		(4)								0		
0_2	0.00	0.02	3	0.00	6	0.02	3	0.00	6	6		
				(6)						0		
0_3	0.20	0.02	1	0.01	1	0.00	7	0.00	7	7		
		(1)	−0.18	(1)	−0.19	(5)				0		
0_4	0.40	0.40	6	0.30	8	0.20	8	0.00	22	22		
		(1)		(1)	−0.10	(3)		(17)		0		
	b_j	6		8		8		17				
			0		0		0		0	0		
	ϵ_j											
	ρ_j											

APPENDIX

TABLEAU 1

		D_1		D_2		D_3		D_4					
	u_i \ v_j	0.0		0.0		0.0		0.0		a_i	x_i^1	δ_i	γ_i
O_1	0.0	0.00	4	0.01	2	0.02	2	0.00	4	4			
		(4)									0		
O_2	0.0	0.02	3	0.00	6	0.02	3	0.00	6	6			
				(6)							0		
O_3	0.0	0.02	1	0.01	1	0.00	(7)	0.00	7	7			
						7					0		
O_4	0.0	0.40	6	0.30	8	0.20	8	0.00	22	22			
								(17)			5	5	4
	R_j	6		8		8		17					
	X_{aj}	2		2		1		0		5			
	ϵ_j							5					
	ρ_j							4					

$h = 0.20$

TABLEAU 2

		D_1		D_2		D_3		D_4					
	u_i \ v_j	0.0		0.0		0.0		−0.20		a_i	x_s^i	δ_i	γ_i
0_1	0.0	0.0	4	0.01	2	0.02	2	0.00	4	4	0		
		④											
0_2	0.0	0.02	3	0.00	⑥	0.02	3	0.00	6	6	0		
				6									
0_3	0.0	0.02	1	0.01	1	0.00	7	0.00	7	7	0	4	3
						⑦							
0_4	0.20	0.40	6	0.30	8	0.20	8	0.00	22	22	4	4	4
						①		⑰					
	R_j	6		8		8		17					
	x_{sj}		2		2		0		0	4			
	ϵ_j					4		4					
	ρ_j					4		4					

$h = 0.01$

TABLEAU 3

		D_1		D_2		D_3		D_4				
	u_i \ v_j	0.0		0.0		−0.01		−0.21		a_i / x_s^i	δ_i	γ_i
O_1	0.0	0.0	4	0.01	2	0.02	2	0.00	4	4		
		(4)								0		
O_2	0.0	0.02	3	0.00	6	0.02	3	0.00	6	6		
				(6)						0		
O_3	0.01	0.02	1	0.01	1	0.00	7	0.00	7	7		
				(1)	−0.01	(6)				0	1	3
O_4	0.21	0.04	6	0.30	8	0.20	8	0.00	22	22		
						(2)		(17)		3	3	4
	R_j	6		8		8		17				
	x_{sj}		2		1		0		0	3		
	ϵ_j					3		3				
	ρ_j					4		4				

$h=0.01$

TABLEAU 4

		D_1		D_2		D_3		D_4		a_i			
	u_i \ v_j	0.00		0.0		−0.02		−0.22			x_s^i	δ_i	γ_i
0_1	0.00	0.0	4	0.01	2	0.02	2	0.00	4	4			
		④									0		
0_2	0.00	0.02	3	0.00	6	0.02	3	0.00	6	6			
				⑥							0		
0_3	0.02	0.02	1	0.01	1	0.00	7	0.00	7	7			
		①	−0.08	①	−0.09	⑤					0	1	3
0_4	0.22	0.40	6	0.30	8	0.20	8	0.00	22	22			
						③		⑰			2	1	4
	R_j	6		8		8		17					
	x_{sj}		1		1		0		0	2			
	ϵ_j					1		1					
	ρ_j					4		4					

$h = 0.08$

TABLEAU 5

		D_1		D_2		D_3		D_4					
	u_i \ v_j	0.00		−0.10		−0.10		−0.30		a_i	x_s^i	δ_i	γ_i
O_1	0.00	0.0	4	0.01	2	0.02	2	0.00	4	4			
		(4)									0		
O_2	0.00	0.02	3	0.00	6	0.02	3	0.00	6	6			
				(6)							0		
O_3	0.10	0.02	1	0.01	1	0.00	7	0.00	7	7			
		(1)	−0.18	(1)	−0.19	(5)					0	1	2
O_4	0.30	0.40	6	0.30	8	0.20	8	0.00	22	22			
				(1)	−0.10	(3)		(17)			1	1	4
R_j		6		8		8		17					
x_{sj}					0		0		0	1			
ϵ_j				1		1		1					
ρ_j				4		4		4					

$h = 0.10$

TABLEAU 6

		D_1		D_2		D_3		D_4					
	u_i \ v_j	0.00		0.00		−0.20		−0.40		a_i	x_s^i	δ_i	γ_i
O_1	0.00	0.0	4	0.01	2	0.02	2	0.00	4	4			
		4									0		
O_2	0.00	0.02	3	0.00	6	0.02	0	0.00	6	6			
				6							0		
O_3	0.20	0.02	1	0.01	1	0.00	7	0.00	7	7			
		1		1		5							
O_4	0.40	0.40	6	0.30	8	0.20	8	0.00	22	22			
		1		1		3		17			0		
	R_j	6		8		8		17					
	x_{sj}		0		0		0		0	0			
	ϵ_j												
	ρ_j												

REFERENCES

[1] Fishburn, Peter, *Decision and Value Theory* (John Wiley and Sons, Inc., 1964).

[2] Hadley, G., *Linear Programming* (Addison-Wesley Publishing Company, Inc., 1962).

[3] Hadley, G., *Nonlinear and Dynamic Programming* (Addison-Wesley Publishing Company, Inc., 1964).

[4] Wagner, Harvey M., *Principles of Operations Research* (Prentice-Hall, Inc., 1969).

ON MAX-MIN PROBLEMS

Kailash C. Kapur

Department of Industrial Engineering and Operations Research
Wayne State University
Detroit, Michigan

ABSTRACT

Necessary and sufficient conditions for max-min problems are given here. In addition to characterization of directional derivatives of the relevant functions, subdifferentiability set for such functions is characterized.

INTRODUCTION

Let $f(x, y)$ be a real valued function of x and y, where $x \epsilon X \subset E^n$ and y belongs to some compact set Y of a topological space. Let us assume that $f(x, y)$ and its partial derivatives with respect to x are continuous in x and y taken together. Let:

$$\phi(x) = \underset{y \epsilon Y}{\text{Min}} f(x, y). \tag{1}$$

Then, the max-min problem is

$$\underset{x \epsilon X}{\text{Max}}\ \phi(x) = \underset{x \epsilon X}{\text{Max}}\ \underset{y \epsilon Y}{\text{Min}} f(x, y). \tag{2}$$

Max-min problems arise very often in system sciences and cybernetics, operations research, economics, game theory, optimal control theory, and duality in mathematical programming. If Y is a finite index set $\{1, 2, \ldots p\}$, then $\phi(x)$ is the scalar function whose value for each x is the least among the p values of f, which may be denoted by $f_1(x), f_2(x), \ldots, f_p(x)$. Hence, problem (2) is to find the maximum of this minimum. This is known as a minimum component maximum problem [5]. Similarly, in control theory and design of systems, we have to consider the problem of minimizing the maximum error or deviations from the desired values of the parameters of the system [7]. For applications in economics, game theory and allocation problems in defense analysis, see Danskin [3, 4] and for max-min problems which arise in duality theory in mathematical programming and control theory see Lasdon [6] and Luenberger [7, 8].

The objective of this paper is to give some necessary and sufficient conditions for optimality for max-min problems which are generalizations of the results due to Danskin [3, 4].

OPTIMALITY CONDITIONS

It is well known that the function ϕ in (2) is not differentiable, but has directional derivatives. A necessary condition for local optimality at a point x° is that the directional derivatives be nonpositive in every admissible direction. An element $\gamma \epsilon E^n$ will be called an admissible direction at a point $x^\circ \epsilon X$ if

there exists a sequence $\{\gamma^n\}$, $\gamma^n \epsilon E^n$, and a scalar sequence $\{a_n\}$, such that

$$\text{a)}\ (x^\circ + a_n\gamma^n)\epsilon X$$

$$\text{b)}\ \gamma^n \to \gamma$$

$$\text{c)}\ a_n > 0 \text{ and } a_n \to 0.$$

The set of all admissible directions is a closed cone and denoted by $C(X, x^\circ)$. For a detailed discussion, see [1, 4, 10].

Let

$$Y(x^\circ) = \{y \epsilon Y \mid f(x^\circ, y) = \phi(x^\circ)\}. \tag{3}$$

Then, the directional derivation of $\phi(x)$ at $x^\circ \epsilon X$ in the direction $\gamma, \|\gamma\| = 1$, is given [3] by

$$D_\gamma \phi(x^\circ) = \operatorname*{Min}_{y \epsilon Y(x^\circ)} \left[\sum_{j=1}^{n} \gamma_j \frac{\partial f(x, y)}{\partial x_j} \right] = \operatorname*{Min}_{y \epsilon Y(x^\circ)} [\gamma, \nabla f(x, y)] \cdot \tag{4}$$

$D_\gamma \phi(x)$ is continuous in x and γ and convex in γ.

LEMMA 1: If x° is a solution for (2), then

$$D_\gamma \phi(x^\circ) \leqq 0 \qquad \text{for all } \gamma \epsilon C(X, x^\circ). \tag{5}$$

PROOF: For $x^\circ + a_n\gamma^n \epsilon X$,

$$\phi(x^\circ + a_n\gamma^n) = \phi(x^\circ) + D_{\gamma n}\phi(x^\circ) \cdot a_n + 0(a_n)$$

$$\leqq \phi(x^\circ),$$

and hence $D_{\gamma n}\phi(x^\circ) \cdot a_n + 0(a_n) \leqq 0$.

When dividing by a_n and taking the limit, the proof follows. Q.E.D.

By using (4), (5) can be written as

$$\operatorname*{Max\text{-}Min}_{\|\gamma\|=1 \ y \epsilon Y(x^\circ)} [\gamma, \nabla f(x, y)] \leqq 0$$

or

$$\operatorname*{Max\text{-}Min}_{\|\gamma\| \leqq 0 \ y \epsilon Y(x^\circ)} [\gamma, \nabla f(x, Y)] = 0.$$

The sufficient condition, which is easy to prove, is given by Lemma 2.

LEMMA 2: If ϕ is concave and X is convex, then (5) is also sufficient for x° to be a solution of

(2). (5) in this case can also be written as

$$\underset{x\in X}{\text{Max-Min}}\ \underset{y\in Y(x^\circ)}{}\ [(x-x^\circ), f(x, y)] = 0.$$

Though function ϕ is not differentiable, it does have subgradients, and we develop an expression for them here.

A vector $\gamma\epsilon E^n$ is said to be subgradient of a concave function ϕ at a point x° if

$$\phi(x) - \phi(x^\circ) \leqq [(x-x^\circ), \gamma] \qquad \text{for all } x.$$

The set of all subgradients of ϕ at a point x° is called the subdifferential of ϕ and denoted by $\partial\phi(x^\circ)$, which is a closed convex set. In [9], it is proven that

$$D_\gamma\phi(x) = \text{Inf}\ [\gamma\cdot\bar{\gamma} | \bar{\gamma}\epsilon\partial\phi(x)]. \tag{6}$$

Let $W(x^\circ)$ denote the set of vectors $\nabla f(x^\circ, y)$ with $y\epsilon Y(x^\circ)$ and let $\hat{W}(x^\circ)$ be the convex hull of $W(x^\circ)$. Then, $\hat{W}(x^\circ)$ is a compact convex set.

THEOREM 1: $\partial\phi(x^\circ) = \hat{W}(x^\circ)$.

PROOF: When $y\epsilon Y(x^\circ)$, $\phi(x^\circ) = f(x^\circ, y)$ and

Hence $$\phi(x) \leqq f(x, y) \qquad \text{for all } x.$$

$$\begin{aligned}\phi(x) - \phi(x^\circ) &\leqq f(x, y) - f(x^\circ, y).\\ &\leqq [(x-x^\circ), \nabla f(x^\circ, y)]\end{aligned}$$

because f is concave in x. Hence, $\nabla f(x^\circ, y)$ is a subgradient of ϕ and $W(x^\circ)\subset\partial\phi(x^\circ)$. Hence, $\hat{W}(x^\circ)\subset\partial\phi(x^\circ)$ because $\partial\phi(x^\circ)$ is a closed convex set. To prove $\hat{W}(x^\circ)\supset\partial\phi(x^\circ)$, let $\bar{\gamma}\epsilon\partial\phi(x^\circ)$ and $\bar{\gamma}\notin\hat{W}(x^\circ)$. Hence, by separating hyperplane theorem [7], there exists a hyperplane γ^* strictly separating $\bar{\gamma}$ and $W(x^\circ)$, because $\hat{W}(x^\circ)$ is compact convex set.

Thus

$$\gamma^*\cdot\bar{\gamma} < \gamma^*\cdot\gamma \qquad \text{for all } \gamma\epsilon\hat{W}(x^\circ)$$

and using (6),

$$D_\gamma{}^*\phi(x^\circ) \leqq \gamma^*\cdot\bar{\gamma} < \underset{\gamma\epsilon\hat{W}(x^\circ)}{\text{Min}}\ \gamma^*\gamma = D_\gamma{}^*\phi(x^\circ),$$

which is impossible. Hence, $\hat{W}(x^\circ)\supset\partial\phi(x^\circ)$ and $W(x^\circ) = \partial\phi(x^\circ)$. Q.E.D.

Thus, by the above theorem, $\gamma \epsilon E^n$ is a subgradient of ϕ at x° if, and only if,

$$\gamma = \sum_{i=1}^{n+1} \lambda_i \nabla f(x^\circ, y_i), \tag{7}$$

where

$$\sum_{i=1}^{n+1} \lambda_i = 1, \qquad \lambda i \geqslant 0 \text{ and } y_i \epsilon Y(x^\circ) \qquad \text{for all } i.$$

The above follows from Caratheodory's Theorem [9] which states that a point in the convex hull of a set in E^n can be represented by a convex combination of at the most $(n+1)$ points of the set.

For a general optimization problem, let the set $X \subset E^n$ be given by

$$X = \{x \mid g_i(x) \leqq 0, \quad i = 1, \ldots, m \text{ and } r_i(x^\circ) = 0, \quad i = 1, \ldots k\}, \tag{8}$$

where g_i and r_i are differentiable functions.

Let L_0 be the subspace spanned by $\nabla r_i(x^\circ)$, $i = 1, \ldots, k$ and $L_0^\perp$ be the orthogonal complement of L_0. Also, let

$$C_{x^\circ} = \{x \mid [x, \nabla g_i(x^\circ)] \leqq 0, \qquad \text{for all } i \text{ such that } g_i(x^\circ) = 0\}.$$

For the necessary conditions, let at a point x°, the following constraint qualification be satisfied:

$$(C_{x^\circ} \cap L_0^\perp) = C(X, x^\circ). \tag{9}$$

For details about generalized constraint qualification see [2, 10].

THEOREM 2: Let at a point x°, the constraint qualification given by (9) is satisfied. Then if x° is an optimal solution of problem (2) there is some nonnegative vector λ, $\lambda_i \geqq 0$, $\sum_{i=1}^{n+1} \lambda_i = 1$, such that

$$\sum_{i=1}^{n+1} \lambda_i \nabla f(x^\circ, y_i) = \sum_{i=1}^{m} u_i g_i(x^\circ) + \sum_{i=1}^{k} v_i r_i(x^\circ)$$

$$u_i g_i(x^\circ) = 0, \qquad i = 1, \ldots, m$$

$$u_i \geqq 0, \qquad v_i \text{ unrestricted and } y_i \epsilon Y(x^\circ).$$

If f is concave function for all $y \epsilon Y$ and the set X is convex, then the above conditions are also sufficient for x° to be a solution of problem (2).

PROOF: By slight generalization of the method given in [1], it can be proved that (using Theorem 1 and Equation (9)):

$$(C_{x^0} \cap L_0^\perp)^* \cap \partial\phi(x^0) \neq \phi,$$

where * denotes the polar or dual cone [7] of the cone $(C_{x^0} \cap L_0^{\perp})$. Hence, for $\gamma \epsilon (C_{x^0} \cap L_0^{\perp})^* \cap \partial \phi(x^0)$, we have

$$\gamma = \sum_{i=1}^{n+1} \lambda_i \nabla f(x^0, y_i)$$

and $\gamma \epsilon (C_{x^0} \cap L_0^{\perp})^*$ which implies that,

$$[\gamma, \nabla g_i(x^0)] \leqq 0 \qquad \text{for all } i \text{ such that } g_i(x^0) = 0$$

$$[\gamma, \nabla r_i(x^0)] \leqq 0 \qquad \text{for all } i = 1, \ldots, k$$

$$[\gamma, -\nabla r_i(x^0)] \leqq 0 \qquad \text{for all } i = 1, \ldots, k.$$

Hence, by Farka's Lemma, we have

$$\sum_{i=1}^{n+1} \lambda_i \nabla f(x^0, y_i) = \sum_i u_i \nabla g_i(x^0) + \sum_{i=1}^{k} (v_{i1} - v_{i2}) \nabla r_i(x^0),$$

where $u_i \geqq 0$ and $v_{i1}, v_{i2} \geqq 0$; and let $u_i = 0$ when $g_i(x^0) < 0$, we have

$$\sum_{i=1}^{n+1} \lambda_i \nabla f(x^0, y_i) = \sum_{i=1}^{m} u_i \nabla g_i(x^0) + \sum_{i=1}^{k} v_i \nabla r_i(x^0)$$

$$u_i g_i(x^0) = 0$$

$u_i \geqq 0$ and v_i unrestricted.

Thus, the above theorem shows how max-min problems can be reduced to a problem of single function optimization problem where at the most $(n+1)$ functions are to be considered and they are added together by weights λ_i. Hence, necessary and sufficient conditions for problem (2) are equivalent to the conditions for

(10)
$$\operatorname*{Max}_{x \epsilon X} \sum_{i=1}^{n+1} \lambda_i f(x, y_i),$$

$$\text{where } y_i \epsilon Y(x), \sum_{i=1}^{n+1} \lambda_i = 1 \text{ and } \lambda_i \geqq 0.$$

CONCLUSIONS

The optimality conditions given above can also be generalized for max-min problems in infinite dimensional spaces. The characterization of the subgradient can aid the development of computational methods for such problems.

REFERENCES

[1] Bram, J., "The Lagrange Multiplier Theorem for Max-Min with Several Constraints," *J. SIAM Applied Mathematics*, Vol. 14, No. 4 (1966), pp. 665–667.

[2] Canon, M., C. Cullum, and E. Polak, "Constrained Minimization Problems in Finite Dimensional Spaces," SIAM Journal Control *4*, 367–389 (1969).

[3] Danskin, J. M., "The Theory of Max-Min with Applications," *J. SIAM Applied Mathematics*, Vol. 14, No. 4 (1966), pp. 641–664.

[4] Danskin, J. M., *The Theory of Max-Min* (Springer-Verlag, New York, Inc., 1967).

[5] Kuhn, H. W., and A. W. Tucker, "Nonlinear Programming," *Proceedings of Second Berkeley Symposium* (University of California Press, Berkeley, 1951), pp. 481–492.

[6] Lasdon, L. S., "Duality and Decomposition in Mathematical Programming," *IEEE Transactions on Systems Science and Cybernetics*, Vol. SSC–4, No. 2 (1968), pp. 86–100.

[7] Luenberger, D. G., *Optimization by Vector Space Methods* (John Wiley and Sons, New York, 1969).

[8] Luenberger, D. G., "Convex Programming and Duality in Normed Space," *IEEE Transactions on Systems Science and Cybernetics*, Vol. SSC–4, No. 2 (1968), pp. 182–188.

[9] Rockafellar, R. T., *Convex Analysis* (Princeton University Press, Princeton, N.J., 1970).

[10] Varaiya, P., "Nonlinear Programming in Banach Space," *SIAM Journal of Applied Mathematics 15*, 284–293 (1967).

A THEORY OF IDEAL LINEAR WEIGHTS FOR HETEROGENEOUS COMBAT FORCES*

David R. Howes

U.S. Army Concepts Analysis Agency

and

Robert M. Thrall

Rice University

ABSTRACT

Detailed combat simulations can produce effectiveness tables which measure the effectiveness of each weapon class on one side of an engagement, battle, or campaign to each weapon class on the other. Effectiveness tables may also be constructed in other ways.

This paper assumes that effectiveness tables are given and shows how to construct from them a system of weapon weights each of which is a weighted average of the effects of a given weapon against each of the enemy's weapons. These weights utilize the Perron-Frobenius theory of eigenvectors of nonnegative matrices. Methods of calculation are discussed and some interpretations are given for both the irreducible and reducible cases.

INTRODUCTION

In conducting military operations research, analysts frequently make use of indices of force effectiveness which are intended to measure the contribution of some force component to the overall power of a military force in some hypothetical military conflict. An example of such an index is the "Firepower Potential" which has been used in a number of U.S. Army analyses as a measure of force strength.† In the alternative considered here, indices are derived from inter-weapon effective matrices (tables) such as might emerge from a detailed combat simulation or from other sources (see for example, [1], [2], [3], [4], [15], [21], [22]).

When such tables are given it is possible to construct from them a system of weapon weights each of which is a weighted average of the effects of a given weapon against each of the enemy's weapons. This paper will describe the construction of such weights.

1. Effectiveness Matrices

Weapon effectiveness may be considered a function of casualty-production which lies in depriving the enemy of the value of weapons lost (cf. [13]). Therefore, it is appropriate to consider numbers which measure the killing power of each weapon against each opposing weapon. An *effectiveness matrix* may be regarded as a table whose entries are these killing powers or relative effectivenesses.

*The original version of this paper appears as Part B of chapter 2 of [26] and was presented at the 11th U.S. Army Operations Research Symposium, May 1972. The authors wish to thank the referee for many helpful suggestions.

†Several references on current procedures are cited below; some others (e.g., [11], [16], [28]) are included among the references without having been cited in the text. A full exposition of past efforts at constructing indices of effectiveness would require access to many classified or otherwise unavailable sources and would go far beyond the scope and purpose of the present paper.

More precisely, consider a combat situation between two opponents, Blue and Red. We suppose that Blue has m classes of weapons and consider the Blue *force vector*

$$U_B = \begin{bmatrix} u_{1B} \\ \cdot \\ \cdot \\ \cdot \\ u_{mB} \end{bmatrix} \tag{1.1}$$

where u_{1B} is the number of Blue weapons of class 1, . . ., u_{mB} is the number of Blue weapons of Blue class m. Similarly, suppose that Red has n classes of weapons and that

$$U_R = \begin{bmatrix} u_{1R} \\ \cdot \\ \cdot \\ \cdot \\ u_{nR} \end{bmatrix} \tag{1.2}$$

is the Red force vector.

In the discussion which follows, it is assumed that the Blue and Red vectors of weights are to be derived in some way from certain interweapon effectiveness matrices; however, there are precedents constructing weight vectors based directly on other considerations. For example, various military operations research organizations (i.e., CORG, RAC, STAG) have from time to time constructed weight vectors based on a consensus of military judgement, individuals being asked to score lists of weapons of types of military unit. Other systems of weights have been based on such considerations as World War II casualties to personnel materiel or on the average damage radii observed during proving ground tests of ammunition (see, for example, [6] and [20]).

The effectiveness matrix concept is connected with Lanchester-type theory of combat in section 5.

We wish to find Blue and Red *weight vectors*

$$W_B = \begin{bmatrix} w_{1B} \\ \cdot \\ \cdot \\ \cdot \\ w_{mB} \end{bmatrix}, \qquad W_R = \begin{bmatrix} w_{1R} \\ \cdot \\ \cdot \\ \cdot \\ w_{nR} \end{bmatrix} \tag{1.3}$$

such that the linear combinations

$$S(B) = w_{1B}u_{1B} + \ . \ . \ . \ + w_{mB}u_{mB} = W_B^T U_B \tag{1.4}$$

and

$$S(R) = w_{1R}u_{1R} + \ . \ . \ . \ + w_{nR}u_{nR} = W_R^T U_R \tag{1.5}$$

are *good* measures of the respective overall strengths of Blue and Red. Then the fraction

$$T = S(B)/S(R) \tag{1.6}$$

can be used as an index of the relative strengths.

A Blue-vs-Red effectiveness matrix M_{BR} is a matrix (table) having m rows and n columns where the element $m_{BR}(i, j)$ measures the effectiveness (killing power) of a single weapon of Blue class i against Red weapon class j. Similarly a Red-vs-Blue effectiveness matrix

$$M_{RB} = [m_{RB}(j, i)] \tag{1.7}$$

has n rows and m columns and, inversely, $m_{RB}(j, i)$ measures the effectiveness of a single Red weapon of class j against Blue weapon class i. The numbers $m_{BR}(i, j)$ and $m_{RB}(j, i)$ may be positive or zero, but, by definition, cannot be negative.

For example, suppose that $m = n = 2$, that both Red and Blue weapon class one are infantry weapons and that both Red and Blue weapon class two are artillery weapons. Then the effectiveness matrices

$$M^1_{BR} = \begin{bmatrix} 0.5 & 0 \\ 0.7 & 0.2 \end{bmatrix}, \quad M^1_{RB} = \begin{bmatrix} 0.6 & 0 \\ 0.6 & 0.1 \end{bmatrix} \tag{1.8}$$

would describe a situation in which (1) in infantry combat Red was more effective than Blue (0.6 vs 0.5), (2) neither infantry could harm the enemy artillery, and (3) the Blue artillery is superior to the Red artillery, and (4) each artillery battery has a positive effectiveness against its counterpart.

The effectiveness matrices

$$M^2_{BR} = \begin{bmatrix} 0.5 & 0.1 \\ 0.7 & 0.2 \end{bmatrix}, \quad M^2_{RB} = \begin{bmatrix} 0.6 & 0.2 \\ 0.6 & 0.1 \end{bmatrix} \tag{1.9}$$

would describe a change which gave each infantry capability against the opposing artillery.

The matrices

$$M^3_{BR} = \begin{bmatrix} 0.5 & 0 \\ 0.7 & 0.8 \end{bmatrix}, \quad M^3_{RB} = \begin{bmatrix} 0.6 & 0 \\ 0.6 & 0.5 \end{bmatrix} \tag{1.10}$$

would describe a different type of change in which the artillery attritions are substantially increased.

If we assume that the artillery units are either concealed or out of each other's range then we could have effectiveness matrices

$$M^4_{BR} = \begin{bmatrix} 0.5 & 0 \\ 0.7 & 0 \end{bmatrix}, \quad M^4_{RB} = \begin{bmatrix} 0.6 & 0 \\ 0.6 & 0 \end{bmatrix} \tag{1.11}$$

2. Ideal Linear Weights

We turn next to consideration of suitable weight vectors, W_B and W_R. These should be derived in

some reasonable way from the corresponding effectiveness matrices, M_{BR} and M_{RB}.

For example, one could simply let W_B be the average of the columns of M_{BR}. Using M^1_{BR} and M^1_{RB} this would give

$$W^1_B=\frac{1}{2}\begin{bmatrix}0.5+0\\0.7+0.2\end{bmatrix}=\begin{bmatrix}0.25\\0.45\end{bmatrix},\quad \text{and } W^1_R=\begin{bmatrix}0.3\\0.35\end{bmatrix}; \tag{2.1}$$

similarly from M^2_{BR} and M^2_{RB} we would obtain

$$W^2_B=\begin{bmatrix}0.3\\0.45\end{bmatrix},\quad W^2_R=\begin{bmatrix}0.4\\0.35\end{bmatrix}. \tag{2.2}$$

This naive approach has the advantage of simplicity, but lacks credibility since it places equal emphasis on effectiveness against enemy infantry and artillery whereas one of these might be considered much more dangerous than the other.

The naive approach places equal weight on each column. A more general procedure is to select as weights nonnegative numbers which add to one. Thus in example 2, if we consider enemy artillery to be twice as important a target as enemy infantry, we would choose weights 1/3, 2/3 and get

$$W^2_B=\begin{bmatrix}1/3(0.5)+2/3(0.1)\\1/3(0.7)+2/3(0.2)\end{bmatrix}=1/3\begin{bmatrix}0.7\\1.1\end{bmatrix}.$$

A vector with nonnegative elements that sum to one is called a *probability vector.* Then the more general procedure would consist of selecting two probability vectors

$$Z_B=\begin{bmatrix}z_{B1}\\.\\.\\.\\z_{Bm}\end{bmatrix},\quad Z_R=\begin{bmatrix}z_{R1}\\.\\.\\.\\z_{Rn}\end{bmatrix} \tag{2.3}$$

and then defining the linear weights by

$$W_B=M_{BR}Z_R,\qquad W_R=M_{RB}Z_B. \tag{2.4}$$

We observe (i) that (2.4) gives each weighting factor w_{iB} as a weighted average (probability combination) of the effectiveness numbers corresponding to the ith Blue weapon type, and (ii) that the same weighted average is used for all i. A still more general procedure would be to permit a different weighted average for each i; this would replace (2.4) by

$$w_{iB}=\sum_j m_{BR}(i,j)\,z_R(i,j),\qquad w_{jR}=\sum_i m_{RB}(j,i)\,z_B(j,i), \tag{2.5}$$

where all columns of the matrices Z_R and Z_B are probability vectors.

Returning now to (2.4) the next step is selection of Z_B and Z_R. In the naive approach we took

$$Z_B=\frac{1}{m}E_m=\frac{1}{m}\begin{bmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{bmatrix}, \qquad Z_R=\frac{1}{n}E_n=\frac{1}{n}\begin{bmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{bmatrix} \tag{2.6}$$

Here (and later) we use the symbol E_p to represent the column vector consisting of p ones, e.g.,

$$E_3=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}.$$

A second, somewhat more reasonable selection is

$$Z_B=M_{BR}E_n/\gamma_B, \qquad Z_R=M_{RB}E_m/\gamma_R, \tag{2.7}$$

where

$$\gamma_B=E_n^TM_{RB}E_m=\sum_{i,j}m_{RB}(j,i), \qquad \gamma_R=E_m^TM_{BR}E_n=\sum_{i,j}m_{BR}(i,j)$$

then

$$W_B=M_{BR}M_{RB}E_m/\gamma_R, \qquad W_R=M_{RB}M_{BR}E_n/\gamma_B. \tag{2.8}$$

In Example 2 this gives

$$Z_B^2=\begin{bmatrix}0.6\\ 0.9\end{bmatrix}\Big/1.5, \quad Z_R^2=\begin{bmatrix}0.8\\ 0.7\end{bmatrix}\Big/1.5 \tag{2.9}$$

and

$$W_B^2=M_{BR}^2Z_R^2=\begin{bmatrix}0.47\\ 0.70\end{bmatrix}\Big/1.5, \quad W_R^2=\begin{bmatrix}0.54\\ 0.45\end{bmatrix}\Big/1.5. \tag{2.10}$$

In (2.7) the jth component of the averaging vector Z_R is proportional to the sum of all Red effectiveness numbers corresponding to the jth Red type. This tacitly assumes equal importance for all Blue weapon types. Clearly, we could modify (2.7) by selecting any nonnegative linear combination $M_{RB}V_B$ of the columns of M_{RB} and then taking Z_R as the unique probability vector proportional to $M_{RB}V_B$. The *ideal*

linear weights which we next introduce correspond to the choice $V_B = W_B$, $V_R = W_R$.

To motivate this choice we consider the following argument. Suppose that W_R has been determined; this means that relative values for the Red weapon systems are known. Then it seems reasonable to select as Z_R the unique probability vector proportional to W_R. Similar reasoning would apply in selection of Z_B if W_B is given. This line of argument would lead to

$$Z_B = W_B/\alpha_B, \qquad Z_R = W_R/\alpha_R \tag{2.11}$$

where

$$\alpha_B = E_m^T W_B, \qquad \alpha_R = E_n^T W_R.$$

then, we get

$$W_B = M_{BR}Z_R = M_{BR}W_R/\alpha_R, \qquad W_R = M_{RB}Z_B = M_{RB}W_B/\alpha_B, \tag{2.12}$$

and by substituting each of these equations in the other we get

$$W_B = M_{BR}M_{RB}W_B/\alpha_B\alpha_R, \qquad W_R = M_{RB}M_{BR}W_R/\alpha_R\alpha_B. \tag{2.13}$$

Now, let

$$P_B = M_{BR}M_{RB}, \qquad P_R = M_{BR}\ M_{BR}, \qquad \lambda = \alpha_B\alpha_R \tag{2.14}$$

and we have the equations

$$P_B W_B = \lambda W_B, \qquad P_R W_R = \lambda W_R. \tag{2.15}$$

The ideal weights must satisfy these equations and also be nonnegative vectors (and also nonzero). At first glance it might seem that (2.11) and (2.12) involve a circular logic since each of the weights is ultimately (cf. 2.15) defined in terms of itself. However, this is a familiar situation in mathematics and is a characteristic of eigenvalue problems which crop up in a wide variety of mathematical models. In particular, Equations (2.15) are well known in linear algebra. First, they require that λ be an eigenvalue of each of the square matrices P_B $(m \times m)$ and P_R $(n \times n)$ and that W_B, W_R be eigenvectors. Since the effectiveness matrices M_{BR}, M_{RB} have nonnegative elements, the same is true of their products P_B, P_R.

The classical Perron-Frobenius theory of eigenvalues and eigenvectors of nonnegative matrices applies to our situation and guarantees solutions to (2.14) with W_B, W_R nonnegative and λ positive. Moreover, it follows from the general theory of matrices that P_B and P_R have the same nonzero eigenvalues. The pertinent facts from the classical Perron-Frobenius theory can be found (with proofs) in chapter XIII of Gantmacher, Vol. II [10]. This chapter also has a comprehensive bibliography (see also Varga [29]). The original papers by Perron and Frobenius appear, respectively, as References [19] and [8], (see also [9], pp 404–414 and 546–567, [14], [27]).

3. Examples of Ideal Weights

We return to our four examples to illustrate the theory.

EXAMPLE 1.

$$P_B^1 = \begin{bmatrix} 0.30 & 0 \\ 0.54 & 0.02 \end{bmatrix}, \qquad P_R^1 = \begin{bmatrix} 0.30 & 0 \\ 0.37 & 0.02 \end{bmatrix}. \tag{3.1}$$

The eigenvalues for both P_B^1 and P_R^1 are $\lambda_1^1 = 0.30$, $\lambda_2^1 = 0.02$. Then

$$Z_B^1 = \begin{bmatrix} 0.34 \\ 0.66 \end{bmatrix}, \qquad Z_R^1 = \begin{bmatrix} 0.43 \\ 0.57 \end{bmatrix} \tag{3.2}$$

are the unique probability eigenvectors corresponding to λ_1^1. The corresponding weights are

$$W_B^1 = M_{BR}^1 Z_R^1 = \begin{bmatrix} 0.215 \\ 0.415 \end{bmatrix}, \qquad W_R^1 = \begin{bmatrix} 0.204 \\ 0.270 \end{bmatrix} \tag{3.3}$$

$$\alpha_B^1 = 0.63, \qquad \alpha_R^1 = 0.474, \qquad \alpha_B^1 \alpha_R^1 = \lambda_1^1 = 0.3.$$

The second eigenvalue λ_2^1 gives

$$Z_B^{1*} = Z_R^{1*} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad W_B^{1*} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \qquad W_R^{1*} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \tag{3.4}$$

We will see later that this second eigenvalue yields less meaningful weights than the first.

EXAMPLE 2.

$$P_B^2 = \begin{bmatrix} 0.36 & 0.11 \\ 0.54 & 0.16 \end{bmatrix}, \qquad P_R^2 = \begin{bmatrix} 0.44 & 0.10 \\ 0.37 & 0.08 \end{bmatrix}. \tag{3.5}$$

The characteristic equation for both matrices is

$$\lambda^2 - 0.52\lambda - 0.0018 = 0, \tag{3.6}$$

and has as its roots the eigenvalues

$$\lambda_1^2 = 0.5235, \qquad \lambda_2^2 = -0.0035. \tag{3.7}$$

From λ_1^2 we get the unique probability eigenvectors

$$Z_B^2 = \begin{bmatrix} 0.40 \\ 0.60 \end{bmatrix}, \qquad Z_R^2 = \begin{bmatrix} 0.545 \\ 0.455 \end{bmatrix} \tag{3.8}$$

for P_B^2 and P_R^2 respectively.

$$W_B^2 = \begin{bmatrix} 0.32 \\ 0.48 \end{bmatrix}, \qquad W_R^2 = \begin{bmatrix} 0.36 \\ 0.30 \end{bmatrix} \tag{3.9}$$

$$\alpha_B^2 = 0.8, \qquad \alpha_R^2 = 0.66, \qquad \alpha_B^2 \alpha_R^2 = 0.528 \sim \lambda_1^2.$$

EXAMPLE 3.

$$P_B^3 = \begin{bmatrix} 0.3 & 0 \\ 0.9 & 0.4 \end{bmatrix}, \qquad P_R^3 = \begin{bmatrix} 0.30 & 0 \\ 0.65 & 0.40 \end{bmatrix}, \qquad \lambda_1^3 = 0.3, \qquad \lambda_2^3 = 0.4. \tag{3.10}$$

This example differs from Example 1 since this time the second eigenvalue is larger than the first and hence λ_1^3 does not correspond to probability eigenvectors (cf. [10] vol. 2, p. 66). The only probability eigenvectors come from λ_2^3 and are

$$Z_B^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Z_R^3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.11}$$

These give

$$W_B^3 = \begin{bmatrix} 0 \\ 0.8 \end{bmatrix}, \qquad W_R^3 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \qquad \alpha_B^3 = 0.8, \qquad \alpha_R^3 = 0.5, \qquad \alpha_B^3 \alpha_R^3 = 0.4 = \lambda_2^3. \tag{3.12}$$

EXAMPLE 4.

$$P_B^4 = \begin{bmatrix} 0.30 & 0 \\ 0.42 & 0 \end{bmatrix}, \qquad P_R^4 = \begin{bmatrix} 0.30 & 0 \\ 0.30 & 0 \end{bmatrix}, \qquad \lambda_1^4 = 0.3, \qquad \lambda_2^4 = 0. \tag{3.13}$$

This example resembles Example 1 in that the first eigenvalue is larger than the second.

From the first eigenvalue we get

$$Z_B^4 = \begin{bmatrix} 0.42 \\ 0.58 \end{bmatrix}, \qquad Z_R^4 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} \tag{3.14}$$

$$W_B^4 = \begin{bmatrix} 0.25 \\ 0.35 \end{bmatrix}, \qquad W_R^4 = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix}, \qquad \alpha_B^4 = 0.6, \qquad \alpha_R^4 = 0.5, \qquad \alpha_B^4 \alpha_R^4 = 0.3 = \lambda_1^4.$$

The second eigenvalue gives

$$Z_B^{4*} = Z_R^{4*} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.15}$$

$$W_B^4 = W_R^4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \alpha_B^4 = \alpha_R^4 = 0,$$

and thus does not provide useful weighting vectors.

Example 2 illustrates a general class of situations where each Blue weapon system is (at least minimally) effective against each Red one and vice versa. If a square matrix P has positive (not merely nonnegative) elements then it has a unique probability eigenvector Z and the corresponding eigenvalue λ_1 (called the *Perron eigenvalue*) is not only positive, but has the largest absolute value of all the eigenvalues of P. It is then easy to calculate Z and λ_1 by the following sequential process (see [14], pp. 151–152 or [27], p. 250). Let $V_0=E_m$ (where P is $m\times m$), let $\alpha(V_0)=E_m^T V_0=m$, let $Z_0=V_0/\alpha(V_0)$, and proceeding inductively let $V_{i+1}=PZ_i$, let $Z_{i+1}=V_{i+1}/\alpha(V_{i+1})$, $i=1, 2 \ldots$. Then

$$Z=\lim_{i\to\infty} Z_i, \qquad \lambda_1=\lim_{i\to\infty}\alpha(V_{i+1}). \tag{3.16}$$

These results still hold (see [10], vol. 2, p. 80) even if P has some, but not too many, zero elements (i.e., as long as P remains irreducible and primitive).

Indeed, when P_B and P_R are positive, we can use a limiting process to define the ideal weights W_B, W_R.

We can begin with W_R^0 any positive vector (e.g., $W_R^0=E_n$) then in turn set $Z_R^0=W_R^0/\alpha(W_R^0)$, $W_B^0=M_{BR}Z_R^0$, $Z_B^0=W_B^0/\alpha(W_B^0)$, and proceeding inductively

$$W_R^i=M_{RB}Z_B^{i-1}, \qquad Z_R^i=W_R^i/\alpha(W_R^i) \tag{3.17}$$

$$W_B^i=M_{BR}Z_R^i, \qquad Z_B^i=W_B^i/\alpha(W_B^i), \qquad i=1, 2, \ldots .$$

Then the six sequences

$$W_R^i, Z_R^i, W_B^i, Z_B^i, \alpha(W_R^i), \alpha(W_B^i) \tag{3.18}$$

converge, respectively, to

$$W_R, Z_R, W_B, Z_B, \alpha_R, \alpha_B \tag{3.19}$$

where Z_B, Z_R are the unique Perron probability eigenvectors of P_B, P_R, respectively; W_R, W_B are the ideal weights for R, B, respectively; $W_R=\alpha_R Z_R$, $W_B=\alpha_B Z_B$, and $\lambda_1=\alpha_R\alpha_B$ is the Perron eigenvalue for both P_B and P_R.

This approach provides a computationally convenient algorithm for calculating the ideal weights. When m and n exceed two, this approach is clearly preferable to calculating and solving the characteristic equation for P_B or P_R. There are other more refined computational algorithms which are, in general, more efficient than this one. However, a computer program written for this iterative process gave quite satisfactory numerical results for moderate values of m and n. An example involving 40 weapon types converged in nine iterations to an accuracy of 0.0001.

4. Interpretation of Reducibility

Examples 1, 3, 4 illustrate some of the possible effects of zeros in P_B, P_R. All of the P's in these examples are what is called *reducible*. A non-negative square matrix P is said to be *reducible* if it has the form

$$P = \begin{bmatrix} P_1 & 0 \\ P_{21} & P_2 \end{bmatrix}, \tag{4.1}$$

where P_1 and P_2 are square, or more generally, if this form can be obtained by a reordering of the rows followed by the same reordering of the columns.

In our combat context, we encounter reducible matrices when as in Examples 1, 3, 4 there are two classes of weapons on each side and the first class of Blue is totally ineffective against the second class of Red and vice versa.

Let us assume that both P_B and P_R are reducible with P_{B1}, P_{B2}, P_{R1}, P_{R2} all positive, that P_{B1}, P_{R1} have the Perron eigenvalue λ_1, and that P_{B2}, P_{R2} have the Perron eigenvalue λ_2. [These assumptions all hold for Examples 1 and 3.] Then, if we apply our computational algorithm beginning with $W_R^0 = E_n$, the limiting eigenvectors obtained will correspond to the larger eigenvalue.

Thus, in Example 1 we would get W_B^1, W_R^1 and not W_B^{1*}, W_R^{1*}. In Example 3 we would, of course, get W_B^3, W_R^3 and in this case there is no possibility of positive ideal weights.

Moreover, in Example 1 the only way to get the starred vectors would be to start with W_R^0 of the form $\begin{bmatrix} 0 \\ a \end{bmatrix}$, i.e., almost all starting vectors W_R^0 will yield W_B^1, W_R^1. For this reason we choose to limit the term "ideal" to W_B^1, W_R^1.

There is a possible interpretation for the different types of weights found in Examples 1 and 3. In Example 1 the attrition of infantry is so much greater than that of artillery that we visualize one phase of the battle ending when one side has lost all of its infantry even though both sides still have artillery left. However, at that time the starred weights do become relevant for the ensuing artillery duel.

On the other hand, in Example 3 the artillery attrition is more rapid than that of infantry. Moreover, when one side runs out of artillery the remaining infantry forces will ultimately be anihilated by the surviving artillery. Hence a zero weight for infantry is not inappropriate.

Example 4 is much like Example 1 for even though $P_{B2} = P_{R2} = 0$ the larger eigenvalue λ_1^4 still gives a viable ideal weight.

5. Calculation of Effectiveness Matrices and an Application to Lanchester Theory

There are several possible approaches to calculation of the effectiveness matrices. Only one of these will be discussed in the present paper.

A sufficiently detailed combat simulation can be expected to produce loss matrices

$$L_{BR} = [l_{BR}(i, j)], \qquad L_{BR} = [l_{BR}(j, i)], \tag{5.1}$$

where $l_{BR}(i, j)$ is the number of Red weapons of class j lost by action of Blue weapons of class i, etc. Then we may define effectiveness matrices M_{BR}, M_{RB} whose elements are the effectiveness numbers;

$$m_{BR}(i, j) = l_{BR}(i, j)/u_{iB}, \qquad m_{RB}(j, i) = l_{RB}(j, i)/u_{jR} \tag{5.2}$$

where U_B and U_R are as in section 1 (formulas (1.1) and 1.2)).

The u_{iB} and u_{jR} might refer either to the initial Blue and Red strengths, or to certain average strengths during the battle. The choice of an appropriate average would relate to questions not con-

sidered here; however, a simple case of such an average might be $[u_{iR(t=0)}+u_{iR(t=t_1)}]/2$ where t_1 is an arbitrary time chosen as a unit of measurement. The interval $(0, t_1)$ must, of course, not exceed the battle length and should be small enough so that combat losses have not yet changed the character of the encounter.

This procedure has as its main drawbacks (1) that the validity of the results obtained depends on the simulation scenario, on the simulation model, and on the extent of sampling error, (2) that it fails to consider military appurtenances which, although affecting the combat action, do not cause attributable casualties to opposing weapon systems, and (3) that it does not take into account scale factors (i.e., it tacitly assumes that the losses are strictly proportional to the number of weapons in a class).

Effectiveness matrices calculated as above are closely related to the Lanchester parameters appropriate to a heterogenous Lanchester linear system and could be interpreted as estimates of such parameters. Such a system represents an extension of the formulae which F. W. Lanchester [17] used to describe the attrition inflicted on each other by two hostile forces to the case where each force is composed of various subelements. In such cases, each force can be represented as a vector of elements and the (scalar) Lanchester attrition coefficients have as counterparts matrices whose elements describe the interacting effects between the elements. These attrition matrices, if known, could serve as examples of effectiveness matrices as discussed in this paper. Conversely, effectiveness matrices, when based on data from real or simulated combat, might be interpreted as Lanchester parameters as noted above.

The generalization of Lanchester equations to the heterogenous case was explored by Snow [23], then by Dolansky [7], and by Bonder and Farrell [5]. It should be noted that effectiveness matrices may be derived in other ways and also that the statistical problem of parameter estimation from sample data is far more complex than might be suggested by the discussion given here.

Dare and James, in Defense Operational Analysis Establishment Memorandum M7120 have made an analysis based on a Lanchester interpretation with results parallel to those which follow next. In Tab E, Appendix II to Annex L of the TATAWS III study, BAARINC Inc. has based a similar analysis on another interpretation.

More specifically, if we have the Lanchester systems

$$\dot{U}_B=-C_RU_R, \qquad \dot{U}_R=-C_BU_B \tag{5.3}$$

then the (i, j) element $c_R(i, j)$ of C_R represents the effectiveness of R weapon j against B weapon i, i.e.,

$$c_R(i,j)=m_{RB}(j,i).$$

Reasoning similarly for C_B we conclude that

$$C_R=M_{RB}^T, \qquad C_B=M_{BR}^T \tag{5.4}$$

are reasonable choices for the Lanchester coefficient matrices.

Now, differentiating Equation (1.4) with respect to time we get

(5.5)
$$\dot{S}(B) = \mathbf{W}_B^T \dot{U}_B = -\mathbf{W}_B^T M_{RB}^T U_R$$
$$= -(M_{RB}\mathbf{W}_B)^T U_R$$
$$= -(M_{RB}M_{BR}\mathbf{W}_R/\alpha_R)^T U_R$$
$$= -\frac{\lambda}{\alpha_R}\mathbf{W}_R^T U_R = -\alpha_B \mathbf{W}_R^T U_R \qquad (\text{since } \lambda = \alpha_R\alpha_B).$$

Now substituting from (1.5) this gives

(5.6)
$$\dot{S}(B) = -\alpha_B S(R).$$

Similarly, differentiating (1.5) yields

(5.7)
$$\dot{S}(R) = -\alpha_R S(B).$$

Equations (5.6) and (5.7) are the ones obtained by Dare and James. A note of caution is appropriate here. The heterogenous systems (5.3), (5.4) are not valid past the time t^* at which any component of U_R or U_B becomes zero. Although the summarizing homogenous systems (5.6) and (5.7) will in general yield solutions $S(B)$, $S(R)$ which both remain positive far beyond t^*, the attrition-rate coefficients α_B and α_R must be modified whenever the weights $\mathbf{W}_B$ and $\mathbf{W}_R$ change due to the annihilation of a target type (see the discussion of Example 1 in section 4).

6. A Larger Example

An example of extended calculation is given below based on results obtained in a particular detailed war game. No claims are warranted concerning the representativeness of these results, which are dependent on the particular scenario, and the random statistical variation inherent in the game model used. Weapons classes for both sides were the same. They were (following some aggregation of similar type):

1. Small arms
2. Armored personnel carriers
3. Tanks
4. Armed reconaissance vehicles
5. Anti-tank weapons
6. Mortars
7. Artillery

Red forces were in the attack, Blue in the defense.

7 Red Weapons 7 Blue Weapons

Red Effects

(6.1) $$M_{RB} = \begin{bmatrix} 0.0145 & 0.0012 & 0.0000 & 0.0229 & 0.0004 & 0.0000 & 0.0000 \\ 0.0510 & 0.0326 & 0.0000 & 0.0638 & 0.0012 & 0.0048 & 0.0000 \\ 0.1060 & 0.4600 & 0.4540 & 0.4900 & 0.0056 & 0.0515 & 0.0000 \\ 0.4440 & 0.2220 & 0.0000 & 0.4440 & 0.0700 & 0.0000 & 0.0000 \\ 0.0000 & 0.1370 & 0.7400 & 0.2740 & 0.0137 & 0.0000 & 0.0000 \\ 6.1500 & 0.0000 & 0.0000 & 0.0000 & 0.0630 & 0.0740 & 0.0000 \\ 21.0000 & 0.2320 & 0.0750 & 0.2770 & 0.1570 & 0.0800 & 0.1960 \end{bmatrix}$$

Blue Effects

(6.2) $$M_{BR} = \begin{bmatrix} 0.0334 & 0.0028 & 0.0000 & 0.0290 & 0.0004 & 0.0000 & 0.0000 \\ 0.1170 & 0.0940 & 0.0000 & 0.1111 & 0.0045 & 0.0000 & 0.0000 \\ 0.4770 & 2.5300 & 2.0900 & 1.8200 & 0.0730 & 0.0000 & 0.0000 \\ 0.8200 & 0.4730 & 0.0000 & 0.5550 & 0.0008 & 0.0000 & 0.0000 \\ 0.0000 & 2.8300 & 0.5000 & 3.3300 & 0.1860 & 0.1940 & 0.0000 \\ 12.0800 & 0.0000 & 0.0000 & 0.0000 & 0.1580 & 0.1502 & 0.0000 \\ 9.7100 & 0.1220 & 0.1000 & 0.1350 & 0.1180 & 0.0680 & 0.2590 \end{bmatrix}$$

(6.3) $$P_R = \begin{bmatrix} 0.0194 & 0.0121 & 0.0002 & 0.0146 & 0.0001 & 0.0001 & 0.0000 \\ 0.1158 & 0.0368 & 0.0006 & 0.0445 & 0.0012 & 0.0010 & 0.0000 \\ 1.2978 & 1.4398 & 0.9517 & 1.1711 & 0.0448 & 0.0088 & 0.0000 \\ 0.4049 & 0.4302 & 0.0350 & 0.5171 & 0.0146 & 0.0136 & 0.0000 \\ 0.5937 & 2.0535 & 1.5534 & 1.5597 & 0.0574 & 0.0027 & 0.0000 \\ 1.0993 & 0.1955 & 0.0315 & 0.3881 & 0.0259 & 0.0233 & 0.0000 \\ 3.8610 & 0.8696 & 0.2548 & 1.4743 & 0.0801 & 0.0558 & 0.0508 \end{bmatrix}$$

(6.4) $$P_B = \begin{bmatrix} 0.0135 & 0.0066 & 0.0003 & 0.0139 & 0.0021 & 0.0000 & 0.0000 \\ 0.0558 & 0.0285 & 0.0033 & 0.0592 & 0.0080 & 0.0005 & 0.0000 \\ 1.1656 & 1.4585 & 1.0029 & 2.0245 & 0.1433 & 0.1198 & 0.0000 \\ 0.2824 & 0.1397 & 0.0006 & 0.2956 & 0.0398 & 0.0023 & 0.0000 \\ 2.8689 & 1.0870 & 0.3646 & 1.9550 & 0.2541 & 0.0537 & 0.0000 \\ 1.0989 & 0.0361 & 0.1169 & 0.3199 & 0.0165 & 0.0111 & 0.0000 \\ 6.0748 & 0.1679 & 0.1521 & 0.4432 & 0.0606 & 0.0315 & 0.0508 \end{bmatrix}$$

Clearly this is a reducible case with one obvious Perron eigenvalue $\lambda_2 = 0.0508$. Applying seven iterations we find that the other Perron eigenvalue λ_1 has the positive probability eigenvectors.

(6.5) $$Z_{1R} = \begin{bmatrix} 0.00052 \\ 0.00198 \\ 0.30482 \\ 0.03033 \\ 0.48015 \\ 0.03087 \\ 0.15134 \end{bmatrix}, \quad Z_{1B} = \begin{bmatrix} 0.00082 \\ 0.00433 \\ 0.54771 \\ 0.01396 \\ 0.26523 \\ 0.06485 \\ 0.10310 \end{bmatrix}$$

where also

(6.6) $$\alpha_{1R}=0.85983, \qquad \alpha_{1B}=1.33191$$

$$\lambda_1=\alpha_R\alpha_B=1.14522$$

$$W_{1R}=\alpha_{1R}Z_{1R}, \qquad W_{1B}=\alpha_{1B}Z_{1B}.$$

Since λ_1 is much greater than λ_2, the ideal weights obtained from λ_1 may be regarded as being more significant than those obtained from λ_2 as given in (6.7) and (6.8) below:

(6.7) $$Z_{2R}=Z_{2B}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\end{bmatrix}$$

(6.8) $$\alpha_{2R}=0.1960, \qquad \alpha_{2B}=0.2590, \qquad \lambda_2=0.0508,$$

$$W_{2R}=\alpha_{2R}Z_{2R}, \qquad W_{2B}=\alpha_{2B}Z_{2B}.$$

BIBLIOGRAPHY

[1] Barfoot, C. B., "The Attrition-Rate Coefficient, Some Comments on Seth Bonder's Paper and a Suggested Alternative Method," Operations Research *17*, 888–894 (1969).

[2] Bonder, Seth, "A Theory for Weapon System Analysis," *Proc. U.S. Army Operations Research Symposium*, 111–128 (1965).

[3] Bonder, Seth, "The Lanchester Attrition-Rate Coefficient," Operations Research *15*, 221–232 (1967).

[4] Bonder, Seth, "The Mean Lanchester Attrition Rate," Operations Research *18*, 179–181 (1970).

[5] Bonder, S. and R. Farrell, "Development of Models for Defense Systems Planning," SRL 2147, Systems Research Laboratory, University of Michigan, Ann Arbor, Michigan (1970).

[6] Corg, "Measuring Combat Effectiveness," Vol. II, Technical Operations Incorporated Inc. Combat Operations Research Group, Alexandria, Va. (Jan. 1970).

[7] Dolansky, L., "Present State of the Lanchester Theory of Combat," Operations Research *12*, 344–358 (1964).

[8] Frobenius, Georg, "Uber Matrizen aus nicht negativen Elementen," *Sitzungsberichte der Kgl Preussischen Akademie der Wissenschaften zu Berlin* (1912), Berlin, pp. 456–477.

[9] Frobenius, Georg, *Gesammelte Abhandlungen, Band III* (Edited by J-P. Serre), Springer-Verlag, Berlin (1968).

[10] Gantmacher, F. R., *The Theory of Matrices* (Chelsea, 1959), 2 vols.

[11] Grubbs, Frank E. and John H. Shuford, "A New Formulation of Lanchester Combat Theory," Operations Research *21*, 926–941 (1973).

[12] Hayward, P., "The Measurement of Combat Effectiveness," Operations Research *16*, 314–323 (1968).

[13] Hero, "Comparative Analyses of Historical Studies," Historical Evaluation and Research Office, 2223 Wisconsin Avenue, Washington, D.C. (15 Oct. 1964), Annex III–H.

[14] Householder, A., *Principles of Numerical Analysis* (McGraw-Hill, New York, 1953).

[15] Kimbleton, S., "Attrition Rates for Weapons with Markov-Dependent Fire," Operations Research *19*, 698–706 (1971).

[16] Koopman, B. O., "A Study of the Logical Basis of Combat Simulation," Operations Research *18*, 855–882 (1970).

[17] Lanchester, F. W., *Aircraft in Warfare, the Dawn of the Fourth Arm* (Constable, London, 1916).

[18] Morse, Philip M., and George E. Kimball, *Methods of Operations Research* (John Wiley, New York, 1951).

[19] Perron, Oskar, "Zur Theorie der Matrices," *Mathematische Annalen*, Vol. 64 (1907).

[20] RAC–TP–III, "Tacspiel War Game Procedures and Rules of Play," Research Analysis Corp. McLean, Va. (Nov. 1963) (Secret).

[21] Rustagi, J. and R. Laitinen, "Moment Estimation in a Markov-Dependent Firing Distribution," Operations Research *18*, 918–923 (1970).

[22] Rustagi, J. and R. Srivastava, "Parameter Estimation in a Markov Dependent Firing Distribution," Operations Research *16*, 1222–1227 (1968).

[23] Snow, R. N., "Contributions to Lanchester Attrition Theory," Project RAND RA–15078 Douglas Aircraft Co., Santa Monica, Cal. (Apr. 1942).

[24] Shuford, John H., "A New Probability Model for Lanchester's Equations of Combat," Masters Thesis submitted to the George Washington University (Dec. 1971).

[25] Taylor, James G., "A Note on the Solution to Lanchester Type Equations with Variable Coefficients," Operations Research *19*, 709–712 (1971).

[26] Thrall, R. M. and Associates, Final Report to U.S. Army Strategy and Tactics Analysis Group, RMT–200–R4–33 (1 May 1972).

[27] Todd, J. (Editor), *Survey of Numerical Analysis* (McGraw-Hill, New York, 1962).

[28] United States Army Combat Developments Command Report, *Measuring Combat Effectiveness*, by Technical Operations Incorporated, Combat Operations Research Group, Vol. I "Firepower Potential Methodology (U)" (Confidential-NO FORN).

[29] Varga, R., *Matrix Iterative Analysis* (Prentice-Hall, Englewood Cliffs, New Jersey, 1962).

[30] Weiss, H. K., "Lanchester-Type Models of Warfare," Proc. *First International Conference on Operations Research* (Dec. 1957), pp. 82–89.

DECISION RULES FOR ATTACKING TARGETS OF OPPORTUNITY

David V. Mastran

OASD(SA)
Department of Defense

and

Clayton J. Thomas

Hq., USAF

ABSTRACT

Frequently in warfare, a force is required to attack a perishable enemy target system—a target system where the targets are detected seemingly at random, and if not immediately attacked, will shortly escape from detection. A conflicting situation arises when an attack element detects a target of relatively low value and has to decide whether to expend his resources on that particular target or to wait for a more lucrative one, hoping one will be found. This paper provides a decision rule giving the least valued target that should be attacked as well as the resources that should be expended as a function of the attack element's remaining mission time.

I. INTRODUCTION

Frequently in warfare, a military force is required to attack a perishable target system. A perishable target system—in contrast to a fixed target system—is one in which the targets are detected seemingly at random, and if not immediately attacked, shortly escape. These "targets of opportunity" are of varying military value and the attacker can attack only a limited number. The number of bombs, missiles, torpedos, or bullets carried by the attacker and the amount of time available to find, attack, and destroy the target are the limiting factors. A conflicting situation arises, then, when an attack element finds a target of relatively low value and has to decide whether to expend resources on that particular target or to wait for a more lucrative target, hoping one will be found. The items of information necessary to make this decision, as well as the means for manipulating the information mathematically, are the subjects of this paper.

Some hypothetical, yet pertinent, examples of this military situation follow:

1. Army

A combat patrol is waiting in an ambush position for an enemy unit to pass by. The patrol is limited to executing one ambush since its presence will then be exposed. Because of morale and endurance problems, however, the patrol can wait for only 3 days by the trail. If one enemy soldier comes down the trail on the second day, under what conditions should the patrol attack him and forego the opportunity of attacking a larger enemy unit?

2. Air Force

An aircraft or flight of aircraft with limited loiter time and ordnance is sent out to attack enemy truck convoys on the Ho Chi Minh trail. The convoys present themselves seemingly at random, appearing from under the jungle foliage, turning off on hidden roads, and again disappearing. These convoys

are of varying sizes consisting of from 1 to 50 or more trucks, and therefore, are of differing values to the attacker. Given knowledge of the expected number of trucks destroyed for the convoy size attacked, how should the attacker decide which convoys to attack and how much ordnance to expend, as the remaining loiter time decreases?

3. Navy

A conventional submarine carrying 16 torpedos is on patrol in the North Atlantic assigned to a shipping lane along which enemy ships are known to travel. The submarine can stay on patrol for a limited number of days. The targets, ships, vary in value to the submarine commander according to tonnage (or, perhaps, cargo carried if the nature of the cargo can be determined). The submarine commander may leave a detected ship if he believes it is not a lucrative target, or he may delay his decision to fire torpedos and remain with the ship, thereby foregoing the chance of finding other valued ships. What size ships should the commander attack and how many torpedos should be expended as a function of the time the submarine has remaining on station?

In these three examples, there are several elements in common. First is that the targets are perishable (fleeting) and of varying value. Second, the attacker can attack only a relatively small number of targets. And third, there is a limited time within which the attacker must act. There are, in fact, many situations both in and out of warfare that have similar conflicting elements in common. The literature is expanding in the area of opportunity analysis.

II. RECURSIVE RELATIONSHIP: ATTACK ONLY ONE TARGET

The basic approach in solving the problem will be the use of Dynamic Programming. A recursive relationship will be derived to obtain a decision rule that specifies when to attack targets. For simplicity, the assumption will be made initially that each attack element can only attack one target and that all ordnance will be expended in the attack.

Since the targets are assumed to be detected at random, a probability distribution will be defined over the times between target detections. From operational data, there may already be available a probability distribution of target "inter-detection" times; otherwise, one will have to be assumed. T_i denotes the probability that i time intervals separate succeeding detections.

$$\sum_{i=1}^{\infty} T_i = 1. \tag{1}$$

The conditional probability D_i that there is a target detection in the next time interval given the last detection occurred $i-1$ time intervals ago is:

$$D_i = T_i / \left(1 - \sum_{j=1}^{i-1} T_j\right) \qquad \text{for } i \geq 2 \tag{2}$$

and

$$D_1 = T_1.$$

Let $g(v)$ be the probability density function of target values measured in terms of the expected target destruction. The distribution is assumed to be stationary and may be discrete for certain classes of targets. Given a target is detected, then, its value V is a random variable drawn from $g(v)$.

Let $f_n(i)$ be the expected return to the attack element—using an optimum policy—when n time intervals remain till the mission is over and i time intervals have passed since the last target detection.

The decision rule is of the form: "Attack the target when n periods remain if and only if its value V is greater than K_n." K_n is a variable changing over time that describes the least valued target that can be attacked.

Given a target detection, then, the expected return of the attack, for the case in which $g(v)$ is continuous, is

$$f_{n-1}(1)\int_{v\leq K_n} g(v)\,dv+\int_{v\geq K_n} vg(v)\,dv. \tag{3}$$

If no target is detected in the nth remaining time interval, the situation transforms to the state with $n-1$ time intervals remaining and $i+1$ intervals since the last detection, i.e., $f_{n-1}(i+1)$.

Combining the terms and the detection probabilities, we get

$$f_n(i)=\operatorname*{Max}_{K_n}\left\{ D_i\left[\int_{v\leq K_n} f_{n-1}(1)g(v)\,dv+\int_{v\geq K_n} vg(v)\,dv\right]+(1-D_i)\ [f_{n-1}(i+1)]\right\}. \tag{4}$$

Obviously $f_n(i)$ is maximized only when a target is attacked that has a higher value than could be obtained by waiting another time interval and obtaining $f_{n-1}(1)$. Hence $K_n=f_{n-1}(1)$. Thus

$$f_n(i)=D_i\left[\int_{v\leq f_{n-1}(1)} f_{n-1}(1)g(v)\,dv+\int_{v\geq f_{n-1}(1)} vg(v)dv\right]+(1-D_i)\ [f_{n-1}(i+1)], \tag{5}$$

$$f_1(i)=D_i\int_{v\geq 0} vg(v)\,dv, \qquad \text{and } f_0(i)=0. \tag{6}$$

By using (5) and (6), we can find $f_n(i)$ for all n and i.

EXAMPLE 1.

Consider the case of an A—26 aircraft with 2 hours of loiter time being sent out to attack convoys in Laos. For simplicity, the target inter-detection time distribution will be assumed to be geometric with a mean of 10 minutes between detections.

$$D_1=D_2=\ \ldots\ =D_\infty=0.1$$

Additionally, because of the memory-less property of the geometric distribution:

$$f_n(i)=f_n(i+1) \qquad \text{all } i\geq 1. \tag{7}$$

Table 1 shows the expected return V in terms of trucks destroyed or damaged given the convoy size attacked. Also provided is the probability distribution of target values (or convoy sizes), $g(v)$, which are discrete in this example.

TABLE 1

Convoy size=i	V_i	$g(v)=P_r\{V=V_i\}$
1	0.25	0.15
2	0.40	0.17
3	0.55	0.20
4	0.70	0.15
5	0.85	0.10
6	1.00	0.05
7	1.15	0.04
8	1.30	0.03
10	1.60	0.03
15	2.00	0.03
20	2.15	0.03
30	2.25	0.02

Some convoy sizes are missing from the table, not necessarily because the enemy operates only in the convoy sizes shown above, but, in part, because the values of i that are given suffice to represent those that are perceived and reported. The recursive relationship in this discrete example is,

$$f_n=0.1\left[\sum_{v\leq f_{n-1}} f_{n-1}g(v)+\sum_{v>f_{n-1}} vg(v)\right]+0.9f_{n-1} \tag{8}$$

$$f_0(\cdot)=0.$$

The solution is shown in Table 2.

TABLE 2

Loiter time remaining=n (min)	$f_n=K_{n+1}$	Minimum size convoy can attack
10	0.505	3
20	0.749	5
30	0.915	6
40	1.043	7
50	1.149	7
60	1.239	8
70	1.317	8
80	1.386	8
90	1.448	8
100	1.503	8
110	1.553	8
120	1.597	8

Thus, the expected return for the A–26 with 2 hours of loiter using an optimal policy is 1.597 trucks damaged or destroyed. This compares quite favorably with 0.758 trucks damaged or destroyed, which is what the A–26 would expect to achieve by attacking the first convoy detected.

A possible objection to this decision rule is that the enemy would observe that only the large convoys were being attacked and would begin operating in smaller convoys. The rule, however, allows even a one-truck convoy to be attacked provided the remaining loiter time is short enough. Moreover, this change in enemy policy could be expressed in a new $g(v)$.

EXAMPLE 2.

In this extension of Example 1, the dependence of $f_n(i)$ on i—the time since the last target detection—is examined for two different distributions. All values are the same as in Example 1 except for the inter-detection time distribution. Let distribution A be defined by

$$\left.\begin{array}{ll} T_j=0.06, & j=1, 2, \ldots, 15 \\ T_j=0, & j=16, \ldots, 27 \\ T_j=0.1, & j=28 \\ T_j=0, & j=29, 30 \ldots \end{array}\right\} \text{A} \tag{9}$$

and distribution B by

$$\left.\begin{array}{ll} T_j=0.091, & j=1, 2, \ldots, 10 \\ T_j=0.001, & j=11, 12, \ldots, 100 \\ T_j=0 & j=101, 102, \ldots \end{array}\right\} \text{B}. \tag{10}$$

Distributions A and B, though quite different from each other, and very different from the geometric distribution of Example 1, have the same mean value as that of Example 1, namely 10. That is:

$$\sum_{j=1}^{\infty} jT_j = 10. \tag{11}$$

An interesting question is how much larger or smaller than $f_n(1)$ does $f_n(i)$ become as i is varied. For representative values of n, and the two distributions A and B, Table 3 provides the answer.

TABLE 3

n	Distribution A			Distribution B		
	$\min f_n(i)$	$f_n(1)$	$\max f_n(i)$	$\min f_n(i)$	$f_n(1)$	$\max f_n(i)$
10	0.000	0.458	0.784	0.087	0.716	0.873
20	0.745	0.760	0.939	0.195	0.890	1.027
30	0.925	0.949	1.074	0.315	0.998	1.114
40	1.000	1.063	1.164	0.443	1.075	1.179
50	1.143	1.170	1.255	0.577	1.140	1.232
60	1.240	1.261	1.334	0.717	1.198	1.282
70	1.320	1.338	1.402	0.862	1.256	1.332
80	1.388	1.406	1.463	1.013	1.315	1.382
90	1.451	1.467	1.518	1.170	1.376	1.436
100	1.508	1.522	1.567	1.246	1.441	1.494
110	1.558	1.570	1.611	1.308	1.499	1.546
120	1.603	1.614	1.651	1.362	1.548	1.591

From Table 3, the sensitivity of $f_n(i)$ to variations in i is much greater for small n. Just how large n must be before the variations become negligible depends on the distribution. By comparing the $f_n(1)$ with the $f_n(\cdot)$ of the geometric distribution in Example 1, one sees the results are moderately close despite the large differences in the distributions.

The decision rule formulations may have applications other than that of making real time decisions. If, for instance, the question were raised—"What would more loiter time allow in terms of increasing expected trucks damaged or destroyed per sortie?"—an answer can be given directly by plotting $f_n(i)$ versus n. The loiter time of an aircraft can be substantially increased by adding sufficient tanker support or decreasing the ordnance load. By assessing the costs of extending the loiter time with the gains achieved, an "optimum" loiter time can be derived.

The example presumed that the decision making was decentralized; the individual aircraft commander made the decision to attack. The attack element in general, however, is really part of a larger force of attack elements, each being assigned a specific sector to patrol. If the decision-making function were centralized, information on the level of enemy activity could be gathered through several intelligence sources and a better perspective attained on the chances of detecting more valuable targets. The centralized decision-making function would not necessarily dictate to the attack elements where to look for targets within a particular sector, but once a target was detected, would give a go-no-go decision to attack.

III. RECURSIVE RELATIONSHIP: MULTIPLE TARGET ATTACKS POSSIBLE

The recursive relation of the last section is generalized in this section to allow more than one attack. Let $f_n(i, m)$ be the expected return for the attack element using an optimum policy when i time units have elapsed since the last target detection and m ordnance units remain to be expended. If a target of value V is attacked with a salvo of W units of ordnance, $h(V, W)$ will be the expected return. In general, the function h will satisfy several inequalities. For the case where h is differentiable in V, these may be written as:

$$h(v, w) < h(v, w+1) \tag{12}$$

$$\frac{\partial h(v, w)}{\partial v} > 0$$

$$\frac{\partial h(v, w-1)}{\partial v} < \frac{\partial h(v, w)}{\partial v}.$$

More precisely, $h(v, w)$ is a monotonically increasing function in both v and w with the property that the rate of increase of the expected return increases with increasing w.

With only one time unit left in the mission, the attack element will attack any target and expend all ordnance. Thus,

$$f_1(i, W) = D_i \int_{\text{all } v} h(v, W) g(v) dv, \tag{13}$$

where D_i is the conditional probability of detection in the next time interval, given the last detection occurred $i-1$ intervals ago.

The decision rule for this formulation of the problem is: "If, when n time units remain in the mission and m ordnance units are available, the value of the target detected V falls in the half open interval $[K_{n,m}^{W}, K_{n,m}^{W+1})$, then use W units of ordnance in an attack on the target."

By definition $K^0_{n,m}=0$ and $K^{m+1}_{n,m}=\infty$ for all n and m. Notice that the constants $K^w_{n,m}$ depend both on the amount of ordnance available and on the time remaining in the mission. For notational simplicity, however, the subscripts n and m will be omitted when they do not change in the formulations presented.

Assuming an attack takes one unit of time, the recursive relationship is

$$(14)\qquad f_n(i,m)=\max_{\bar{K}}\left\{D_i\sum_{w=0}^{m}\int_{K^w}^{K^{w+1}}[h(v,w)+f_{n-1}(1,m-w)]g(v)dv+(1-D_i)f_{n-1}(i+1,m)\right\},$$

where $\bar{K}$ is the set of $m+2$ tuples——

$$\bar{K}=\{(K^0,K^1,\ldots,K^{m+1}):0\leqslant K^a\leqslant\infty \text{ and } K^a<K^{a+1},\ a=0,1,\ldots,m\} \text{ and } f_n=0 \text{ for } n\leqslant 0.$$

To maximize $f_n(i,m)$ over the set of $m+2$ tuples, differential calculus can be used. $f_{n-1}(i,m)$, $g(v)$, and D_i are independent of $K^w_{n,m}\equiv K^w$.

Selecting any element of the $m+2$ tuple and taking partial derivatives,

$$(15)\qquad \frac{\partial f_n(i,m)}{\partial K^a}=\frac{\partial}{\partial K^a}D_i\int_{K^{a-1}}^{K^a}[h(v,a-1)+f_{n-1}(1,m-a+1)]g(v)dv$$

$$+\frac{\partial}{\partial K^a}D_i\int_{K^a}^{K^{a+1}}[h(v,a)+f_{n-1}(1,m-a)]g(v)\,dv$$

or

$$(16)\qquad \frac{\partial f_n(i,m)}{\partial K^a}=D_ig(K^a)A,$$

where

$$(17)\qquad A=h(K^a,a-1)+f_{n-1}(1,m-a+1)-h(K^a,a)-f_{n-1}(1,m-a).$$

Since $g(K^a)>0$, $A=0$ at a minimum or maximum. To test for a maximum, one calculates

$$(18)\qquad \frac{\partial^2 f_n(i,m)}{\partial K^{a2}}=D_ig(K^a)\frac{\partial A}{\partial K^a}+D_i\frac{\partial g(K^a)}{\partial K^a}A.$$

Since the second term becomes 0 at $A=0$, one has

$$(19)\qquad \frac{\partial^2 f_n(i,m)}{\partial K^{a2}}=D_ig(K^a)\left[\frac{\partial h(K^a,a-1)}{\partial K^a}-\frac{\partial h(K^a,a)}{\partial K^a}\right].$$

The term in brackets is negative because of the properties postulated for $h(v,w)$. Therefore, when $A=0$, one has the maximum value of $f_n(i,m)$. K^a, then, using (17) is the solution to the equation

$$h(K^a, a-1) - h(K^a, a) = f_{n-1}(1, m-a) - f_{n-1}(1, m-a+1). \tag{20}$$

EXAMPLE 3.

For this example, assume a submarine has four torpedos remaining and 100 hours left on station. The single shot hit probability p is assumed to be independent of target value and equal to 0.3.*

The expected return, given that the target value is V and the number of torpedos fired is W, is given by

$$h(V, W) = V[1 - (1-p)^W]. \tag{21}$$

For simplicity, the inter-detection time distribution will be assumed geometric with a mean of 20 hr: $D_i = 0.05$, $i = 1, 2, \ldots, \infty$.

The normalized target values will be assumed to be distributed uniformly between 0 and 10.

$$g(v) = 1/10, \qquad 0 \leq v \leq 10 \tag{22}$$

$$= 0, \qquad \text{otherwise.}$$

The equation yielding K^a, after introducing the functional form of (21), becomes

$$K^a = [f_{n-1}(1, m-a+1) - f_{n-1}(1, m-a)]/p(1-p)^{a-1}. \tag{23}$$

The solution procedure is first to find $K^a_{1,m}$ for all m and a given $f=0$. Then having the $K^a_{1,m}$ find the $f_1(1, m)$. This procedure is repeated, first finding the $K^a_{n,m}$, then the $f_n(1, m)$.

The solution is shown below in Figures 1–4. Figure 5 represents the situation in which the submarine has m torpedos, $m = 1, 4$. Along the ordinate is target value and along the abscissa the time remaining on station. The numbers 0–4 indicate the number of torpedos that should be fired in the specific "regions" of target value and time remaining on station.

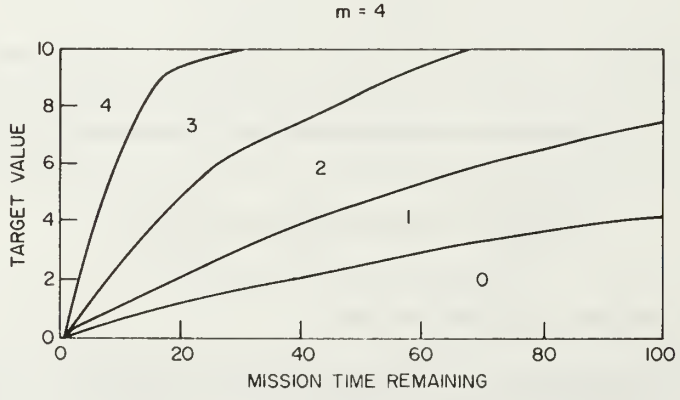


FIGURE 1

*In general, p does not have to be independent of target value.

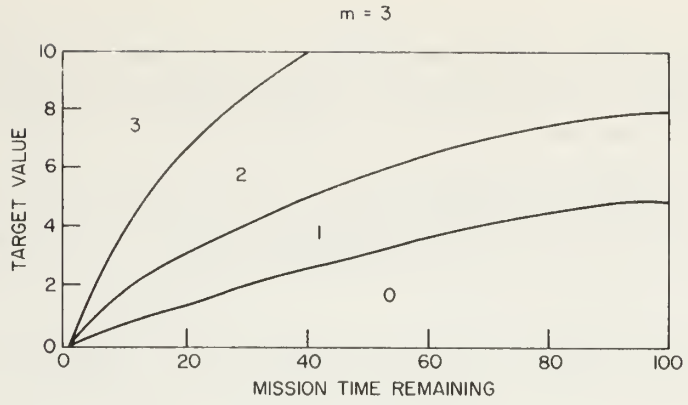


FIGURE 2

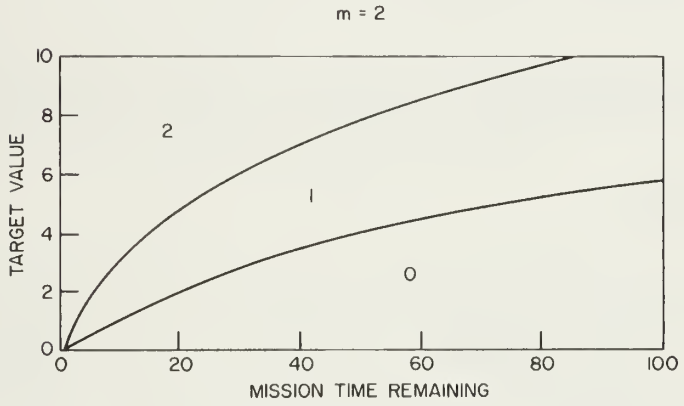


FIGURE 3

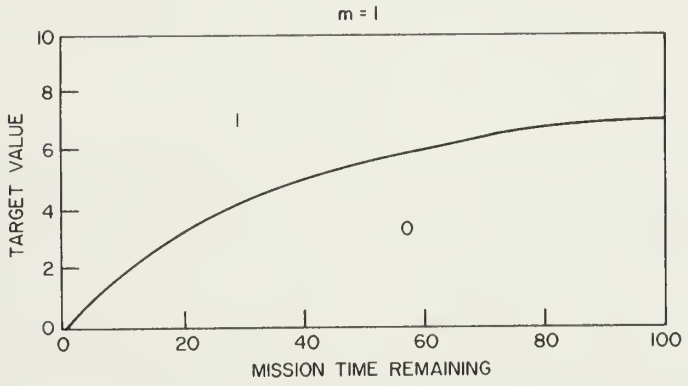


FIGURE 4

If the submarine attacked the first target found with all four torpedos, the expected return would be 3.78. On the other hand, by employing the decision rule just described, the submarine can expect to obtain a return of 6.72, a 78-percent improvement.

EXAMPLE 4.

In the previous example, the submarine commander was assumed to salvo all torpedos. If the commander desired to employ a "shoot-look-shoot" policy, the decision rule and the formulation must be modified to permit the expenditure of less than w units of ordnance, when observations of prior damage so suggest.

Assuming p is the single shot probability of hit, the expected return for a "shoot-look-shoot" policy—given W units of ordnance can be expended and a target of value V is detected—is

$$(24) \quad p[V+f_{n-1}(1, m-1)] + p(1-p)[V+f_{n-1}(1, m-2)] + \ldots + p(1-p)^{W-1}[V+f_{n-1}(1, m-W)] + (1-p)^{W}f_{n-1}(1, m-W).$$

The last term is included since $f_n(1, m-W)$ is returned whether or not the last ordnance unit hit the target. Simplifying the above expression, the return, expressed as $h'(V, W)$ is

$$(25) \quad [1-(1-p)^{W}]V + \sum_{j=1}^{W} p(1-p)^{j-1}f_{n-1}(1, m-j) + (1-p)^{W}f_{n-1}(1, m-W).$$

The recursive relationship is

$$(26) \quad f_n(i, m) = \max_{K} \left\{ D_1 \sum_{w=0}^{m} \int_{K^w}^{K^{w+1}} h'(v, w)g(v)dv + (1-D_i)f_{n-1}(i+1, m) \right\}.$$

Finding the K^a as before by taking partial derivatives,

$$\frac{\partial f_n(i, m)}{\partial K^a} = D_i g(K^a) \cdot A,$$

where

$$(27) \quad A = -p(1-p)^{a-1}K^a + (1-p)^{a-1}f_{n-1}(1, m-a+1) - (1-p)^{a-1}f_{n-1}(1, m-a).$$

Setting $A = 0$ for the maximum yields

$$K^a_{n,m} = [f_{n-1}(1, m-a+1) - f_{n-1}(1, m-a)]/p.$$

The solution for this formulation produces two somewhat surprising results. First is that $K^a_{n,m} = K^{a-1}_{n,m-1}$ for all n and m and $a > 1$. Instead of four figures as in the last example, then, only one need be drawn (Figure 5), since the uppermost region signifies using up to all remaining ordnance units; the next uppermost region, all but one remaining unit of ordnance, etc.

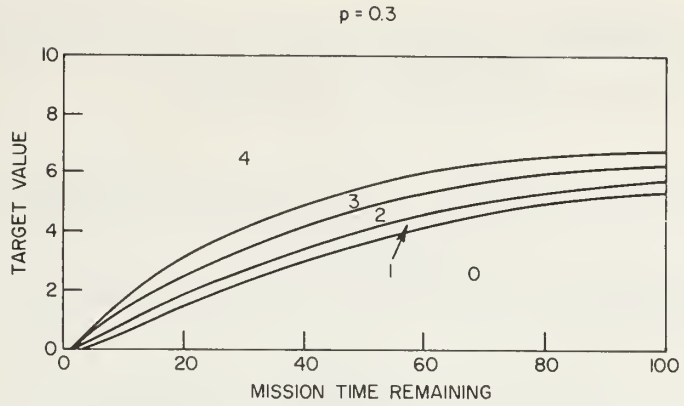


FIGURE 5

The second surprising result is that, $K^m_{n,m}$ is independent of p, the single shot probability of hit; the other values of $K^a_{n,m}$ for $a \neq m$ are dependent on p, however. In fact, as $p \to 0$, $K^1_{n,m} \to K^m_{n,m}$ for all n and m; in effect saying either expend no ordnance or expend up to all remaining ordnance. On the other hand, as $p \to 1$, $K_{n,m} \to 0$ making the decision rule more selective in establishing the number of ordnance units to expend. Figure 6 shows the results for $p = 0.99$.

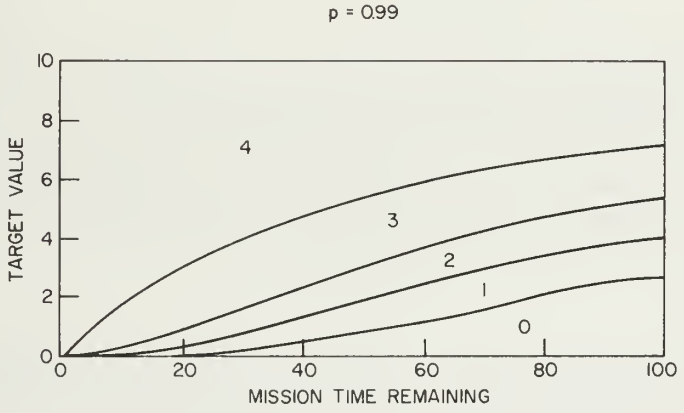


FIGURE 6

In terms of the improvement in expected return for a "shoot-look-shoot" policy, the submarine commander with 100 hours remaining can expect a total return of 7.65 versus 6.72 for the salvo decision rule. This represents 13.8-percent improvement over the salvo rule and a 102-percent improvement over the expected return when salvoing all four torpedos at the first ship detected.

EXAMPLE 5.

In this simple extension to Examples 3 and 4, the target values or expected return will be allowed to change over time. If the mission time is long and the war is expected to be a short intense one, the same target will be "worth" more the earlier it is destroyed. Allowing the expected return to change can be accomplished by discounting the target values as the mission time decreases.

Let

$$h(v, w|n) = h(v, w) \cdot (1 - \alpha^n),$$

where $(1-\alpha^n)$ is the discount term.

Because of the structure of the recursive relationship, the discount term can be factored out of the expression leaving the previous results valid for determining $K^a_{n,m}$.

$$(28) \qquad f_n(i, m) = (1-\alpha^n) \max_{\bar{K}} \left\{ D_i \sum_{w=0}^{m} \int_{K_w}^{K_{w+1}} h(v, w) g(v) dv \right\} + (1-D_i) f_{n-1}(i+1.\ m).$$

By letting $\alpha = 0.95$, the table below compares $K^4_{n,4}$ from Examples 4 and 5.

TABLE 4

n	$K^4_{n,4}$ (Example 4)	$K^4_{n,4}$ (Example 5)
10	1.872	0.431
20	3.266	1.330
30	4.249	2.287
40	4.981	3.163
50	5.546	3.926
60	5.997	4.578
70	6.364	5.132
80	6.670	5.602
90	6.928	6.002
100	7.148	6.343

Notice that by discounting, the decision rule becomes more lenient in allowing attacks using all ordnance. This occurs because the target values are decreasing.

There are other extensions to the formulations than can be incorporated such as allowing the attack time to exceed a unit time interval or changing the detection probabilities after an attack has been executed. These extensions are simple ones and hence will not be addressed. The flexibility of the Dynamic Programming approach is considerable.

IV. CONCLUSIONS

The Dynamic Programming formulations appear to be quite useful in determining what a lucrative target is. To the extent that the distributions required can be estimated or empirically derived, the approach proposed in this paper will be valid. Other considerations such as the threat to the attack element from Anti-Aircraft Artillery, Anti-Submarine Warfare, etc., must be addressed, however. Future work will investigate the effects of attrition on the decision rule.

BIBLIOGRAPHY

[1] Bellman, Richard E., *Dynamic Programming* (Princeton University Press, Princeton, New Jersey, 1957).

[2] Bellman, Richard E. and Stuart E. Dreyfus, *Applied Dynamic Programming* (Princeton University Press, Princeton, New Jersey, 1962).

TARGET SELECTION IN LANCHESTER COMBAT: LINEAR-LAW ATTRITION PROCESS

James G. Taylor*

Department of Operations Research and Administrative Sciences
Naval Postgraduate School
Monterey, California

ABSTRACT

We develop the solution to a simple problem of target selection in Lanchester combat against two enemy force types each of which undergoes a "linear-law" attrition process. In addition to the Pontryagin maximum principle, the theory of singular extremals is required to solve this problem. Our major contribution is to show how to synthesize the optimal target selection policies from the basic optimality conditions. This solution synthesis methodology is applicable to more general dynamic (tactical) allocation problems. For constant attrition-rate coefficients we show that whether or not changes can occur in target priorities depends solely on how survivors are valued and is independent of the type of attrition process.

1. INTRODUCTION

In a recent paper [26] we have presented some elements of a mathematical theory of target selection in dynamic combat situations. We did this through the examination of the structure of the optimal allocation policies for some tactical situations described by Lanchester-type equations of warfare. The purpose of this previous paper was to contrast the structures of the optimal allocation policies for various scenarios. In the present paper we develop results for a prescribed duration battle in which enemy target types undergo a "linear-law" attrition process (see section 3). For reasons of brevity, we had previously [26] just stated these results without justification.

The problem under study is solved by the mathematical theory of optimal control. Its solution, however, requires more than the well-known Pontryagin maximum principle [22]: the theory of singular extremals (see chapter 8 of [4]) must be used to solve it. A brief discussion of the required theory of singular extremals is included in this paper. By an extremal we mean a battle trajectory on which the necessary conditions of optimality are satisfied everywhere in time.

The major contribution of this paper is to show how to synthesize the optimal control in combat against target types which undergo a "linear-law" attrition process. In this case, singular subarcs (see section 2) may be present in the battle trajectory. By the synthesis of optimal control, we mean the explicit determination of the time history of the optimal control from initial to terminal time as a function of the initial state of the system. There is no general method for the synthesis of optimal controls in singular problems [19]; each class of problems possesses its own peculiarities. Hence, an understanding of how to synthesize the optimal control in this elementary problem is particularly important,

*This research was supported by the Office of Naval Research as part of the Foundation Research Program at the Naval Postgraduate School and partially under Project Order No. 2-0150.

since it provides insight for more complex extensions that we have considered in our subsequent researches.

The body of this paper is organized in the following fashion. First, we review that part of the theory of singular extremals which is required for the solution of the problem under study. Next, we present our model and develop the basic necessary conditions of optimality. Then, we discuss how to synthesize extremals using the necessary conditions. Next, we explain how the optimality of the extremal control is demonstrated by the existence of an optimal control and the uniqueness of extremals. Then, we show how to synthesize the solution to our problem. This is done for the two cases of import. Finally, we make some comments about the structure of the optimal target selection policies and extensions.

2. THE THEORY OF SINGULAR EXTREMALS

In an optimal control problem, the maximum principle may fail to determine an optimal trajectory, since the maximization of the Hamiltonian may not lead to a well-defined expression for optimal control [16], [14] (also see chapter 8 of [4]). Singular solutions usually occur when the Hamiltonian is a linear function of the control variables. However, all problems for which the Hamiltonian is a linear function of the control variables do not have singular subarcs in their solution.

The problem that we shall consider has one control variable, and it appears linearly in the Hamiltonian. By a *singular subarc* we denote that part of an optimal trajectory on which the maximum principle cannot be used to determine the control because the coefficient of the control variable in the Hamiltonian is zero* (see pp. 226–227 of [15]). Then the term "singular solution" will be used to refer to any optimal trajectory which contains one or more singular subarcs.

To elaborate further, when the Hamiltonian H is a linear function of the control variable ϕ, then if $\frac{\partial H}{\partial \phi}=0$ for a finite interval of time (or, another way to say this, the coefficient of ϕ vanishes identically for a finite interval of time), then the maximum principle does not determine the control. Observe that in this case *all* feasible values of ϕ maximize the Hamiltonian. When this happens we determine the singular control by requiring that we remain on the singular subarc, i.e., $\frac{\partial H}{\partial \phi}$ remains zero. If $\frac{\partial H}{\partial \phi}$ is to be identically equal to zero for a finite interval of time, then all its derivatives with respect to time must also be equal to zero. We determine the singular control, which keeps the system on the singular subarc, by considering as many of the time derivatives of $\frac{\partial H}{\partial \phi}$ as are required for the control variable ϕ to appear explicitly so that it may be determined from an algebraic equation. Thus, in general we consider

$$0=\frac{\partial H}{\partial \phi}=\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right)=\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right)=\ldots. \tag{1}$$

For the problem at hand, an explicit expression for the singular control is obtained from the equation

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right)=0 \tag{2}$$

*This statement, of course, depends upon the form of Hamiltonian employed (see section 5).

by use of the conditions $\frac{\partial H}{\partial \phi}=\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right)=0$ and the canonical equations (i.e., both state and adjoint system).

We must further check to make sure that we can get a maximum return (in the case when we wish to maximize the criterion functional) from use of the candidate singular subarc. The following condition (generalized Legendre-Clebsch condition) is necessary for a singular subarc to yield a maximum return

$$(3) \qquad (-1)^k \frac{\partial}{\partial \phi}\left\{\frac{d^{2k}}{dt^{2k}}\left(\frac{\partial H}{\partial \phi}\right)\right\} \leq 0.$$

It is obtained by examining the negative semidefiniteness of the second variation for a special class of explicitly defined control variations [17]. For the problem at hand, it suffices to consider the generalized Legendre-Clebsch condition with $k=1$. Recently, Jacobson [10] discovered a new necessary condition for optimality on singular subarcs. This condition is not readily checked, however, for the problem at hand, since the details of application are extremely messy.

It should be emphasized that the generalized Legendre-Clebsch condition (3) is merely a necessary condition of optimality so that even if it is satisfied on a singular subarc, we are not guaranteed that the criterion functional is maximized. Very recently, Jacobson [11] gave sufficient conditions for non-negativity of the second variation in singular and nonsingular control problems. These yield conditions analogous to the well-known no-conjugate-point condition (see pp. 181–184 of [4]) for singular subarcs and have led to necessary and sufficient conditions of optimality for singular control problems [23] (see also [13]). In this paper we will, however, use a different approach to prove that we have found the optimal control (see section 7 below). The interested reader can find comprehensive bibliographies on the singular control problem in [11], [12], and [17].

3. THE MODEL

We consider the following prescribed duration battle

$$\underset{\phi(t)}{\text{maximize}} \{ry(T)-px_1(T)-qx_2(T)\} \text{ with } T_1 \text{ specified,}$$

subject to:

$$\frac{dx_1}{dt}=-\phi a_1 x_1 y,$$

$$\frac{dx_2}{dt}=-(1-\phi)a_2 x_2 y,$$

$$(4) \qquad \frac{dy}{dt}=-b_1 x_1-b_2 x_2,$$

with initial conditions

$$x_1(t=0)=x_1^0, \qquad x_2(t=0)=x_2^0, \qquad y(t=0)=y_0,$$

and

$$x_1, x_2, y \geq 0, \qquad 0 \leq \phi \leq 1, \qquad \text{and} \qquad T \leq T_1,$$

where all symbols are defined in section 4 and the battle termination conditions are elaborated upon below.

The battle lasts for $0 \leq t \leq T_1$ unless, of course, one side or the other is annihilated before T_1. To be more precise, the battle terminates under one of the following three conditions:

$$\begin{aligned} &TC_1 \pm x_1(T = x_2(T) = 0 \quad \text{and} \quad T \leq T_1, \\ &TC_2 : y(T) = 0 \quad \text{and} \quad T \leq T_1, \\ &TC_3 : T = T_1, \end{aligned} \tag{5}$$

where T denotes the time at which the battle ends.

In the above problem (4) x_1, x_2, and y are called *state* variables, while ϕ is called a *control* (or decision) variable. A constraint, such as $x_1 \geq 0$ is called a state variable inequality constraint (SVIC) and requires special treatment (see chapter 6 of [22]). In other words, the well-known maximum principle* (as presented in chapters 1–3 of [22]) requires modification in problems with SVIC's (see also chapter 3 of [4]). Moreover, McIntyre and Paiewonsky [21] remarked in 1967 that "the optimal control problem with state space constraints does not appear to be well understood." It has been the personal experience of the author that this is even true in the applications literature today [24].

Using the corner conditions from the theory of state variable inequality constraints (SVIC's) (see [14], [21], or pp. 125–126 of [4]), it will be shown below that an optimal policy can only lead to the following extremal terminal states

$$\begin{aligned} &E_1 : x_1(T) = x_2(T) = 0, \quad y(T) > 0, \quad \text{and } T \leq T_1 \text{ with } \dot{x}_1(t) > 0 \text{ for } t < T \text{ and } x_2(t) > 0 \text{ for } t < T, \\ &E_2 : x_1(T) > 0, x_2(T) > 0, y(T) = 0, \quad \text{and } T \leq T_1, \\ &E_3 : x_1(T) > 0, x_2(T) > 0, y(T) > 0, \quad \text{and } T = T_1. \end{aligned} \tag{6}$$

In other words, for $t < T$ we have $x_1^*(t) > 0$, $x_2^*(t) > 0$, and $y^*(t) > 0$ so that no SVIC is active for $t < T$ when an optimal policy is followed. Hence, the SVIC's are essentially ignored in developing the solution to the problem at hand (except, of course, for the establishment of the fact that for an optimal policy $x_1^*(t) > 0$ when $t < T$, etc.).

In previous papers [26], [27] we have described the basic scenario under consideration and also the circumstances which lead to a "linear-law" attrition process. As before [26], we refer to attrition as being a "linear-law" process when the casualty rate is proportional to the product of the number of enemy firers and remaining targets. We have also discussed at length the structure of the optimal target selection policies and its implications for military tactics previously [26].

Since the state and adjoint equations do not readily yield an analytic solution for quantities such as $x_1(t)$ or $p_3(t)$, it has not been possible to obtain explicit expressions for certain model parameters.

---

* There is a difference in sign between the version of the maximum principle used by Pontryagin et al [22] and an equivalent version commonly used in the control theory literature of this country (see p. 108 of [4]).

Moreover, the author has not been able to derive explicit expressions for optimal trajectories or controls in all cases. However, one can still discuss all the qualitative characteristics of the optimal allocation policy.

4. NOTATION

The symbols which are used in this paper are defined as follows:

a_1, a_2, b_1, b_2 = constant attrition-rate coefficients,

E_i for $i = 1, 2, 3$ = the ith extremal terminal state as defined by (6)

H = Hamiltonian function,

J = criterion functional $= ry(T) - px_1(T) - qx_2(T)$,

k = constant of proportionality,

L = singular "surface" defined by $a_1b_1x_1 = a_2b_2x_2$,

L' = line (with equation $a_1px_1 = a_2qx_2$) in description of solution to problem,

$\mathrm{L} = \mathrm{L}(\phi, \lambda)$ = Lagrangian function,

$P(t=T) = (x_1(t=T), x_2(t=T))$,

p, q, r = utilities assigned per unit of surviving X_1, X_2, and Y forces, respectively,

$p_i(t)$ for $i = 1, 2, 3$ = dual variable corresponding to $x_i(t)$ (with $x_3(t) = \mathrm{y}(t)$),

t = time after beginning of battle,

t_c = time of occurrence of "corner,"

t_e = time of entry to constrained subarc,

$t_l = T - \tau_l$ = time which separates Phase I of the battle from Phase II as described in section 6,

t_s = time of entry to singular subarc,

$t_1 = T - \tau_1$ = time of first switch in extremal tactics,

T = total time for the battle,

T_1 = maximum possible duration for battle, i.e., $T \leqslant T_1$,

TC_i for $i = 1, 2, 3$ = battle termination conditions as defined by (5),

$v(t) = a_1x_1(-p_1(t)) - a_2x_2(-p_2(t))$,

W = Bellman's optimal value function,

x_1, x_2, y = combatant force levels; with initial values x_1^0, x_2^0, y_0,

δ = a positive constant,

δ_1 = a positive constant,

λ = a positive constant,

Λ = Lagrange multiplier defined in (A. 15),

ϕ = fraction of Y-fire directed at X_1,

τ = "backwards time" from the end of the battle; defined by $\tau = T - t$, i.e., the time remaining before the end of battle,

$\tau_l = T - t_l$ = "backwards time" which separates Phase I of the battle from Phase II as described in section 6.,

τ_1, τ_2, etc. = "backwards time" of the first, second, etc., switch in extremal tactics

5. DEVELOPMENT OF BASIC NECESSARY CONDITIONS OF OPTIMALITY

We now develop the basic necessary conditions of optimality which hold on extremals. In appendices A and B it is shown that it is nonoptimal to have $x_1(t) = 0$ with $x_2(t) > 0$. Considering the battle

termination conditions (5), it suffices to consider here only the case in which $x_1(t) > 0$, $x_2(t) > 0$, and $y(t) > 0$ for $t < T$. Under these circumstances the Hamiltonian for the above problem (4) is given by [4]

$$H(t, x_i, p_i, \phi) = (-p_1 a_1 x_1 y + p_2 a_2 x_2 y)\phi + \{-p_2 a_2 x_2 y - p_3(b_1 x_1 + b_2 x_2)\}, \tag{7}$$

where $p_i(t)$ for $i = 1, 2, 3$ are the dual variables corresponding to the state variables x_1, x_2, $x_3 = y$ (see [26], [27] for a discussion of the military significance of these variables). The maximum principle leads to the following nonsingular optimal control (when there is only one extremal (see section 7.)

$$\phi^*(t) = \begin{cases} 0 & \text{for } p_2 a_2 x_2 < p_1 a_1 x_1, \\ 1 & \text{for } p_2 a_2 x_2 > p_1 a_1 x_1. \end{cases} \tag{8}$$

The adjoint system of differential equations for the dual variables is given by

$$\begin{aligned} \frac{dp_1}{dt} &= \phi^* a_1 y p_1 + b_1 p_3, \\ \frac{dp_2}{dt} &= (1 - \phi^*) a_2 y p_2 + b_2 p_3, \\ \frac{dp_3}{dt} &= \phi^* a_1 x_1 p_1 + (1 - \phi^*)\, a_2 x_2 p_2. \end{aligned} \tag{9}$$

The boundary conditions for the dual variables at $t = T$ are discussed below.

Additionally, at a corner which occurs at an interior point of the state space (i.e., $x_1 > 0$, $x_2 > 0$, $y > 0$) the following well-known corner conditions must hold [4]

$$p_i(t_c^-) = p_i(t_c^+) \qquad \text{for } i = 1, 2, 3, \tag{10}$$

and

$$H^*(t_c^-) = H^*(t_c^+), \tag{11}$$

where t_c^- denotes the time just before the corner, t_c^+ denotes the time just after the corner, and $H^*(t_c^-)$ denotes $H(t = t_c^-, x_i^*, p_i, \phi^*)$. The reader should recall that a "corner" is a place of discontinuity of the slope of the state space trajectory. Thus, a corner is a place where

$$\frac{d\vec{x}}{dt}(t_c^-) \neq \frac{d\vec{x}}{dt}(t_c^+), \tag{12}$$

where

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ y(t) \end{pmatrix}.$$

On a singular subarc $\frac{\partial H}{\partial \phi} = 0$ for a finite interval of time. From (7) we readily compute that

$$\frac{\partial H}{\partial \phi} = y(p_2 a_2 x_2 - p_1 a_1 x_1), \tag{13}$$

$$\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = -(b_1 x_1 + b_2 x_2)(p_2 a_2 x_2 - p_1 a_1 x_1) + p_3 y(a_2 b_2 x_2 - a_1 b_1 x_1), \tag{14}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = &-(p_2 a_2 x_2 - p_1 a_1 x_1)\frac{d}{dt}(b_1 x_1 + b_2 x_2) - 2p_3(b_1 x_1 + b_2 x_2)(a_2 b_2 x_2 - a_1 b_1 x_1) \\ &+ y(a_2 b_2 x_2 - a_1 b_1 x_1)\frac{dp_3}{dt} + y^2 p_3\{\phi[a_1(a_1 b_1 x_1) + a_2(a_2 b_2 x_2)] - a_2(a_2 b_2 x_2)\},\end{aligned} \tag{15}$$

where both the state equations (4) and the adjoint equations (9) have been used in the development of (14) and (15). Considering (13), the requirement that $\frac{\partial H}{\partial \phi} = 0$ yields the *first condition for a singular subarc*

$$p_1 a_1 x_1 = p_2 a_2 x_2. \tag{16}$$

Considering (14) and (16), the requirement that $\frac{d}{dt}\left(\frac{\partial H}{\partial \phi}\right) = 0$ on a subarc on which $\frac{\partial H}{\partial \phi} = 0$ yields the *second condition for a singular subarc*

$$a_1 b_1 x_1 = a_2 b_2 x_2. \tag{17}$$

On a subarc on which the first and second conditions for a singular subarc hold (i.e., (16) and (17)), we additionally require that

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = 0, \tag{18}$$

so that evaluating (15) on a subarc on which both (16) and (17) hold we obtain

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial \phi}\right) = y^2 p_3 a_1 b_1 x_1\{\phi^*(a_1 + a_2) - a_2\} = 0, \tag{19}$$

which readily yields the *singular control* which is required to keep the system on the singular subarc

$$\phi^* = \frac{a_2}{a_1 + a_2}. \tag{20}$$

To check the generalized Legendre-Clebsch condition on the singular subarc, we further differentiate

(15) with respect to ϕ and evaluate the result on the subarc on which (16) and (17) hold. Doing this, we find that

$$\frac{\partial}{\partial\phi}\left\{\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\phi}\right)\right\} = y^2 p_3(t)\{(a_1)^2 b_1 x_1 + (a_2)^2 b_2 x_2\} > 0, \tag{21}$$

since as is shown below $p_3(t) > 0$ for all $t < T$. Thus, the necessary condition is met for the singular path to be optimal.

Considering the battle termination conditions (5), the fact that an optimal policy results in x_1, x_2, $y > 0$ for $t < T$ yields that an optimal policy can only lead to the extremal terminal states (6). For each of these, the boundary conditions for the dual variables at $t = T$ are given in Table I. These results were obtained as follows. When $x_1(T) > 0$, $x_2(T) > 0$, or $y(T) > 0$, the determination of the value of the corresponding dual variable at $t = T$ is routine (see [4]). When $y(T) = 0$ with $T < T_1$, consideration of (7) and the transversality condition

$$H(t = T, x_i^*, p_i, \phi^*) = 0, \tag{22}$$

yields that

$$p_3(t = T) = 0. \tag{23}$$

When $x_1(T) = x_2(T) = 0$ with $T < T_1$, it is clear by (8) and (20) that we must be on a singular subarc for $T - \gamma \leq t \leq T$ (where $\gamma > 0$). Hence, by (16) and (17) we have that for $t < T$ (since $x_1(t)$ and $x_2(t) > 0$ for $t < T$)

$$\frac{p_1(t)}{p_2(t)} = \frac{b_1}{b_2} \qquad \text{for } T - \gamma \leq t \leq T. \tag{24}$$

TABLE I. *Boundary Conditions of Dual Variables for Various Extremal Terminal States*

Terminal state E_1: $x_1(T) = x_2(T) = 0$, $y(T) > 0$, and $T \leq T_1$ with $x_1(t) > 0$ for $t < T$ and $x_2(t) > 0$ for $t < T$. $p_1(t = T) = -kb_1$, $p_2(t = T) = -kb_2$, $p_3(t = T) = r$ with $k > 0$.
Terminal state E_2: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) = 0$, and $T \leq T_1$. $p_1(t = T) = -p$, $p_2(t = T) = -q$, $p_3(t = T) = 0$.
Terminal state E_3: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) > 0$, and $T = T_1$. $p_1(t = T) = -p$, $p_2(t = T) = -q$, $p_3(t = T) = r$.

Thus

$$\lim_{t \leq T} p_i(t) = -kb_i \qquad \text{for } i = 1, 2. \tag{25}$$

Since T is unspecified, we also have

$$H(t, x_i^*, p_i, \phi^*) = 0, \tag{26}$$

so that on the singular subarc on which (16), (17), and (20) hold, we have that

$$H(t, x_i^*, p_i, \phi^*) = -p_1 a_1 x_1 y - p_3(b_1 x_1 + b_2 x_2). \tag{27}$$

Combination of (17), (26), and (27) then yields that

$$p_3(t) = -\left\{\frac{a_1 a_2 y}{b_1(a_1 + a_2)}\right\} p_1(t). \tag{28}$$

Now recalling that for $y(T) > 0$ we have $p_3(t = T) = r > 0$, by continuity of p_3 we obtain that $p_3(t) > 0$ for t close to T. Hence, by (28) we have

$$p_1(t) < 0 \qquad \text{for all } t \text{ close to } T, \tag{29}$$

since the quantity in brackets in (28) is positive. By (25) and (29), we have

$$p_i(t = T) = -kb_i \quad \text{for } i = 1, 2, \text{ with } k > 0. \tag{30}$$

To establish the fact that $p_3(t) > 0$ for all $t < T$ we proceed as follows. It is well-known [2], [18], [22] that one can make the identification

$$p_3(t) = \frac{\partial W}{\partial y(t)}, \tag{31}$$

except for certain manifolds of discontinuity (switching surfaces) [2] (see also [3]) where W is not differentiable (see p. 73 of [22]). In (31) $W = W(t, x_1, x_2, y)$ denotes Bellman's optimal value function [1] (i.e. $W(t, x_1, x_2, y)$ denotes the return o tained when the system state at t is (x_1, x_2, y) and an optimal policy $\phi^*(s)$ is followed for $t \leq s \leq T$). It is well-known that the corners of broken extremals lie on such manifolds of discontinuity [2], [3]. Hence, at a point of the state space which lies on an extremal *not* at a corner, the W function is differentiable, and we can make the identification (31). Furthermore, considering the state equations (4), at such a point it is clear that we must have

$$p_3(t) > 0, \tag{32}$$

since addition of another Y combatant at t can only result in an increase in return to Y when an optimal

policy $\phi^*(s)$ is followed for $t \leq s \leq T$. At a corner we can invoke (10) to establish (32) everywhere for $t < T$.

6. SYNTHESIS OF EXTREMAL CONTROLS

By the synthesis of the extremal control we mean the explicit determination of the time history of the extremal control from initial to terminal time. This is done by combining the extremal control with integration of the state and adjoint systems of equations. The maximum principle has yielded the nonsingular control (8) while (1) has yielded the singular control (20) as well as the first and second conditions for a singular subarc, (16) and (17). It should be noted that this extremal control (both nonsingular and also singular) is a function of both the state and adjoint variables.

The above yields a two point boundary value problem: the state and adjoint systems of equations (i.e., (4) and (9)) are to be solved using the extremal control with (initial) conditions for the state variables given at $t = 0$ and (boundary) conditions given for the dual variables at $t = T$ (see Table I). Additionally, we have established in section 5 that there are only three extremal terminal states. For each of these, we may start at the end of the battle at $t = T$ (where boundary values for the dual variables are known) and obtain the time history of the dual variables by a backwards integration of the adjoint system of differential equations (and also state equations) combined with the extremal control. The corner conditions (10) and (11) are also used in doing this.

By this process we can trace extremals backwards from each terminal state of battle. This backwards synthesis process (see also [7] and [9] for further discussions) is carried out so that an extremal path leads from the terminal conditions to the initial conditions for the case under study. In other words, the backwards synthesis is carried out in such a way as to guarantee satisfaction of the initial conditions.

In synthesizing the extremal course of battle (backwards from the end of the prescribed duration battle) it is convenient to introduce

$$v(t) = -a_1 p_1 x_1 + a_2 p_2 x_2. \tag{33}$$

By (8) and (20) the extremal control may be expressed in terms of $v(t)$ as

$$\phi^*(t) = \begin{cases} 1 & \text{for } v(t) > 0, \\ \dfrac{a_2}{a_1 + a_2} & \text{for } v(t) = 0, \\ 0 & \text{for } v(t) < 0. \end{cases} \tag{34}$$

We recall that (17) must also hold on a singular subarc.

Since we develop the solution to this problem by working backwards from the end $t = T$, it is convenient to introduce the "backwards time" variable τ defined by $\tau = T - t$. Observing that $\dfrac{d}{dt} = -\dfrac{d}{d\tau}$ and using both the state equations (4) and the adjoint system (9), we obtain from differentiation of (33) that

$$\frac{dv}{d\tau} = (a_1 b_1 x_1 - a_2 b_2 x_2) p_3. \tag{35}$$

Thus, we see that on a singular subarc on which $v(\tau) = 0$ we also have that $\frac{dv}{d\tau} = 0$. Also, it is sometimes convenient to write (33) as

$$v(\tau) = -\left(\frac{p_2(\tau)}{b_2}\right)\left[\frac{\left(\frac{p_1(\tau)}{p_2(\tau)}\right)}{\left(\frac{b_1}{b_2}\right)}(a_1b_1x_1) - a_2b_2x_2\right]. \tag{36}$$

Let us focus on extremal terminal state E_3:$x_1(T) > 0$, $x_2(T) > 0$, $y(T) > 0$, and $T = T_1$. At the end of battle $t = T$, we have

$$v(\tau = 0) = a_1px_1(t = T) - a_2qx_2(t = T). \tag{37}$$

Taking (36) into consideration, we see that a point on the singular "surface"* $a_1b_1x_1 = a_2b_2x_2$ yields a positive, zero, or negative value for $v(\tau)$ at $\tau = 0$ depending upon whether $\frac{p}{q}$ is greater than, or equal to, or less than $\frac{b_1}{b_2}$. Hence, by (34) a battle trajectory which has reached the singular surface can, in general, only remain on it for a finite interval of time ending at the end of battle (i.e., remain on the singular surface for $0 \leq \tau \leq \delta$ where $\delta > 0$, or equivalently $T - \delta \leq t \leq T$) when $\frac{p}{q} = \frac{b_1}{b_2}$. Thus, in synthesizing extremal trajectories we must consider three cases.

Case (a) $\frac{p}{q} = \frac{b_1}{b_2}$,

Case (b) $\frac{p}{q} > \frac{b_1}{b_2}$,

Case (c) $\frac{p}{q} < \frac{b_1}{b_2}$.

The solution for Cases (a) and (b) has been described by us in a previous paper [26].

If we were to plot in Figure 1 the line L' defined by $a_1px_1 = a_2qx_2$, then it would appear above, on, or below the line L defined by $a_1b_1x_1 = a_2b_2x_2$ depending on whether $\frac{p}{q}$ were greater then, equal to, or less than $\frac{b_1}{b_2}$. This is evident from considering the slopes of these two lines.

---

*We refer to the locus of points such that $a_1b_1x = a_2b_2x_2$ as the singular "surface," since if (16) holds at $t = t_s$ (hence $v(t_s) = 0$) then a trajectory remains on this "surface" for $t_s \leq t \leq t_1$ by use of the singular control (20). The reason for this "singular" behavior is that, as (35) shows, $\frac{dv}{dt} = 0$ so that $v(t)$ remains identically equal to zero for a finite interval of time and hence the maximum principle fails to determine the control.

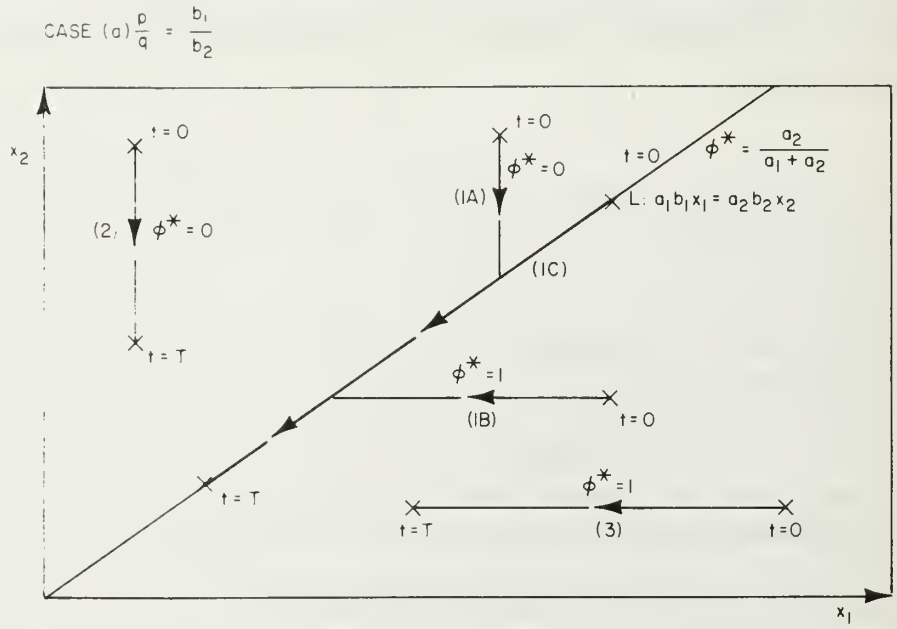


FIGURE 1. Optimal allocation for linear-law attrition process: survivors valued in direct proportion to their attrition-rate coefficients

$$\left(\frac{dx_2}{dx_1}\right)_L = \frac{a_1 b_1}{a_2 b_2}, \qquad \left(\frac{dx_2}{dx_1}\right)_{L'} = \frac{a_1 p}{a_2 p},$$

since, for example,

$$\frac{p}{q} > \frac{b_1}{b_2} \text{ implies that } \left(\frac{dx_2}{dx_1}\right)_{L'} > \left(\frac{dx_2}{dx_1}\right)_L.$$

The significance of the line L' and its relationship to the line L is as follows. The battle is divided into two time phases: Phase I for $0 \leq t \leq t_I = T - \tau_I$ and Phase II for $T - \tau_I = t_I \leq t \leq T$. During Phase I the optimal* target engagement policy at a point in time is determined by the location of the point on the battle trajectory with respect to the line L, which is also the singular "surface." Above L, $\phi^*(t) = 0$; while below L, $\phi^*(t) = 1$. When a battle trajectory reaches L, it remains on the singular surface through use of the singular control $\phi^* = \frac{a_2}{a_1 + a_2}$. During Phase II the optimal target engagement policy is to use $\phi^*(t) = 1$ below L'. It may be shown that it is impossible for a battle trajectory to cross L' during Phase II.

The above results will be developed in two sections below on the synthesis of optimal control. The following relationships readily follow from previously developed results and are required to establish the results of the above paragraph

$$v(\tau = 0) \begin{cases} > 0 \text{ below } L', \\ < 0 \text{ above } L', \end{cases} \tag{38}$$

*The optimality of extremals is discussed in section 7.

so that

$$\phi^*(t=T)=\begin{cases} 1 & \text{for } P(T) \text{ below } L', \\ 0 & \text{for } P(T) \text{ above } L', \end{cases} \tag{39}$$

where $P(t=T)=(x_1(t=T), x_2(t=T))$. We also note that by (35)

$$\frac{dv}{d\tau}(\tau)\begin{cases} >0 \text{ below } L, \\ =0 \text{ on } L, \\ <0 \text{ above } L. \end{cases} \tag{40}$$

7. DETERMINATION OF THE OPTIMAL CONTROL

As the reader is undoubtedly aware, the maximum principle only furnishes necessary conditions of optimality. Thus, it remains to demonstrate the optimality of an extremal trajectory. Two ways in which this may be done are as follows:

(a) show that sufficient conditions of optimality are satisfied on the extremal;

(b) by citing the appropriate existence theorem, show that an optimal control exists to the problem at hand; there are two further subcases: (1) if the extremal is unique, then it is optimal or (2) if the extremal is not unique and only a finite number exist, then the optimal trajectory is determined by considering the finite number of alternatives.

It has not been convenient to take the former approach to the problem at hand. One cannot invoke the sufficient conditions of Funk and Gilbert [8] (which are an extension of the results of Mangasarian [20]), since (translating into the terminology of the present paper) the right hand sides of the state equations are not concave functions of x_1, x_2, y, and ϕ. Additionally, the sufficient conditions of Jacobson [11], [12] and Jacobson and Speyer [13], [23] are not readily checked for the problem at hand.

Hence, the second approach given above is the one that we have taken in this paper. If an extremal is unique (in the sense that only one extremal leads from the initial point in the state space to the terminal surface), then it is optimal and no difficulty exists. However, we have not been able to treat explicitly the case of multiple extremals as we did in [25]. In the Isbell-Marlow fire programming problem a domain of controllability (see [25]) was determined by inequalities involving the three state variables; in the present prescribed duration battle such a determination involves the four variables T_1, x_1^0, x_2^0, y_0. In other words, for the problem at hand we may consider time, t, as an additional state variable. We have not been able to quantitatively determine the optimal control when multiple extremals exist (although we can still give a qualitative discussion) because we could not develop time solutions for $x_1(t), x_2(t)$, or $y(t)$ (see below).

The existence of an optimal control is established by invoking an existence theorem due to Lee and and Markus (see Corollary 2 on p. 262 of [19] (which extends some earlier results of these authors [18])). The key aspects of being able to apply this result are the linearity of the control variable and the uniform boundedness of responses to the controller ϕ. (The latter condition is a consequence of the fact that ϕ is restricted to lie in a compact set and appears linearly in the state equations (4).) When the control variables do not appear linearly, the existence of an optimal control is much more difficult to establish (see, for example, [5] and [6]).

It will be shown (see section 8) that the extremals are unique in Case (a): $\frac{p}{q} = \frac{b_1}{b_2}$. Hence, the extremal control is optimal in this case. However, the uniqueness of extremals has not been established in the other two cases for all regions of the initial force level space. It appears as though extremals may lead from some regions of initial force levels to both E_2 and E_3 (see (6)). We have not been able to make an explicit determination about this because it has not been possible to solve (4) to obtain a "closed-form" solution for, for example, $y(t)$. This latter fact has precluded the analytic computation in general of the return functional, denoted by J, in order to determine under what conditions optimal paths lead to E_2 and E_3 as was done in [25].

It would be convenient to compute the return functional corresponding to a particular control (see [25]). Thus, we would like to express

$$J = ry(T) - px_1(T) - qx_2(T), \tag{41}$$

as

$$J = J(x_1^0, x_2^0, y_0, T_1). \tag{42}$$

Although a state equation is readily obtained for (4) (see Table II), the author has not been able to develop time solutions for $x_1(t)$, $x_2(t)$, and $y(t)$ (also $p_1(t)$, $p_2(t)$, and $p_3(t)$) except for special cases. Hence, the author has not been able to develop (42). Due to the above situation it has not been possible to obtain an explicit expression for the switching time $t_1 = T - \tau_1$. (However, this could be done for the problem studied in [25].) Moreover, in computational studies such quantities may be numerically computed by finite difference methods.

Finally, let us make some observations about the entries shown in Table II. When $\phi(t) = 0$ for $t_1 \leqslant t \leqslant t_2$, the equation for $y(t)$ in 1 becomes

$$y(t) = \left[y^2(t_1) + \frac{2b_1x_1(t_1)}{a_2} ln\left(\frac{x_2(t)}{x_2(t_1)}\right) + \frac{2b_2}{a_2}\{x_2(t) - x_2(t_1)\}\right]^{1/2}. \tag{43}$$

This may be obtained either by direct ingegration of (4) or by applying L' Hospital's theorem to the result for $y(t)$ in 1 of Table II for $0 \leqslant \phi < 1$. A similar expression may be obtained for $y(t)$ when $\phi(t) = 1$ for $t_1 \leqslant t \leqslant t_2$.

8. SOLUTION SYNTHESIS WHEN SURVIVORS VALUED IN PROPORTION TO KILL RATES

For Case (a): $\frac{p}{q} = \frac{b_1}{b_2}$, optimal battle trajectories are shown in Figure 1. Above the line L with equation $a_1b_1x_1 = a_2b_2x_2$ the optimal control is to use $\phi^*(t) = 0$ until this line is encountered. When a trajectory reaches L, the singular control $\phi^* = \frac{a_2}{a_1 + a_2}$ (which keeps the trajectory on L) is used until the end of battle at $t = T$. Below L, $\phi^*(t) = 1$ is used in a similar fashion. To establish these results, we work backwards from each possible type of end point of battle.

First, we trace extremals backwards from each extremal terminal state of battle. We give the complete development for E_3: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) > 0$, and $T = T_1$. At the end of battle $\tau = 0$ equation

TABLE II. *State Equations for Two-on-One Combat.*

1. *When* $0 \leq \phi(t) = \text{constant} < 1$ *for* $t_1 \leq t \leq t_2$

$$x_1(t) = x_1(t_1)\left(\frac{x_2(t)}{x_2(t_1)}\right)^{\frac{\phi a_1}{(1-\phi)a_2}}$$

$$y(t) = \left[y^2(t_1) + \frac{2b_1x_1(t_1)}{a_1}\left\{\left(\frac{x_2(t)}{x_2(t_1)}\right)^{\frac{\phi a_1}{(1-\phi)a_2}} - 1\right\} + \frac{2b_2}{(1-\phi)a_2}\{x_2(t) - x_2(t_1)\}\right]^{1/2}$$

2. *When* $0 < \phi(t) = \text{constant} \leq 1$ *for* $t_1 \leq t \leq t_2$

$$x_2(t) = x_2(t_1)\left(\frac{x_1(t)}{x_1(t_1)}\right)^{\frac{(1-\phi)a_2}{\phi a_1}}$$

$$y(t) = \left[y^2(t_1) + \frac{2b_2x_2(t_1)}{(1-\phi)a_2}\left\{\left(\frac{x_1(t)}{x_1(t_1)}\right)^{\frac{(1-\phi)a_2}{\phi a_1}} - 1\right\} + \frac{2b_1}{\phi a_1}\{x_1(t) - x_1(t_1)\}\right]^{1/2}$$

3. *When* $\phi(t) = \dfrac{a_2}{a_1 + a_2}$ *for* $t_1 \leq t \leq t_2$

$$x_1(t) = \left(\frac{x_1(t_1)}{x_2(t_1)}\right)x_2(t)$$

$$y(t) = \left[y^2(t_1) + 2\left(\frac{a_1 + a_2}{a_1a_2}\right)\left\{\frac{b_1x_1(t_1) + b_2x_2(t_1)}{x_1(t_1)}\right\}\{x_1(t) - x_1(t_1)\}\right]^{1/2}$$

additionally if $a_1b_1x_1 = a_2b_2x_2$, *then*

$$y(t) = \left[y^2(t_1) + 2a_1b_1\left(\frac{a_1 + a_2}{a_1a_2}\right)^2\{x_1(t) - x_1(t_1)\}\right]^{1/2}$$

(36) reduces to

$$(44) \qquad v(\tau = 0) = \left(\frac{q}{b_2}\right)[a_1b_1x_1(t = T) - a_2b_2x_2(t = T)],$$

since we have assumed $\dfrac{p}{q} = \dfrac{b_1}{b_2}$. By (44) we see that there are three cases to consider depending on the sign of the term in square brackets.

CASE (1): $a_1b_1x_1(t = T) = a_2b_2x_2(t = T)$

This corresponds to when the system ends up on the singular subarc. In this case $\phi^*(t = T) = a_2/(a_1 + a_2)$, and for $0 \leq \tau \leq \tau_1 =$ the "backwards time" of the first switch, we use the singular control $\phi^*(\tau) = a_2/(a_1 + a_2)$. Let us note that use of the singular control for $0 \leq \tau \leq \tau_1$ results in $\dfrac{dv}{d\tau} = 0$ so that $v(\tau) = v(\tau = 0) + \displaystyle\int_0^\tau \frac{dv}{d\tau} = 0$. At $t_1 = T - \tau_1$ we switch control, since $x_1(t_1) = x_1^0$ or $x_2(t_1) = x_2^0$.

This yields three further subcases.

SUBCASE (1A): $a_1b_1x_1^0 < a_2b_2x_2^0$

At $t = t_1 > 0$ we have that $a_1b_1x_1^0 = a_2b_2x_2(t_1) < a_2b_2x_2^0$ so that we cannot destroy anymore x_1. Then we use $\phi^*(\tau) = 0$ for $\tau_1 \leq \tau \leq T$. This is consistent since $v(\tau = \tau_1) = 0$ and

$$\frac{dv}{d\tau} = p_3(a_1b_1x_1^0 - a_2b_2x_2) < 0 \qquad \text{for } \tau_1 < \tau \leq T.$$

(Observe that for $\tau_1 < \tau \leq T$, $\frac{dx_2}{d\tau} = a_2x_2y$ so that $x_2(\tau) > x_2(\tau_1)$.) This implies that $v(\tau) < 0$, and hence $\phi^*(t) = 0$ for $0 \leq t \leq t_1 = T - \tau_1$.

SUBCASE (1B): $a_1b_1x_1^0 > a_2b_2x_2^0$

A similar argument readily yields that $\phi^*(t) = 1$ for $0 \leq t \leq t_1$.

SUBCASE (1C): $a_1b_1x_1^0 = a_2b_2x_2^0$

We use $\phi^*(t) = a_2/(a_1 + a_2)$ from the beginning.

CASE (2): $a_1b_1x_1(t = T) < a_2b_2x_2(t = T)$

Since $v(\tau = 0) = \left(\frac{q}{x_2}\right)[a_1b_1x_1 - a_2b_2x_2] < 0$, at the end of battle we have $\phi^*(t = T) = 0$. Hence, for $0 \leq \tau \leq \tau_1 =$ the "backwards time" of the first switch, we use $\phi^*(\tau) = 0$. We work backwards from the end. Since we are above the line L, $\frac{dv}{d\tau} = p_3(a_1b_1x_1 - a_2b_2x_2) < 0$. Hence, $v(\tau) < 0$ for all $\tau \epsilon [0, T]$, and we never do switch. Thus, we have that $\phi^*(t) = 0$ for $0 \leq t \leq T$.

CASE (3): $a_1b_1x_1(t = T) > a_2b_2x_2(t = T)$

A similar argument to that used for Case (2) readily yields that $\phi^*(t) = 1$ for $0 \leq t \leq T$.

The above cases are shown in Figure 1. It should be noted that the above development depends upon the fact that $p_3(t) > 0$ for all t. It should further be noted that, in general, trajectories (1A), (1B), and (1C) will not all terminate in the same point as shown in Figure 1, which was drawn this way for simplicity. Details are similar for extremals leading to E_1 and to E_2 (see (6)), and are, therefore, omitted. (The reader should recall that in these two subcases the boundary conditions for the dual variables are given in Table I.) In this Case (a), the extremals are unique and hence optimal.

9. SOLUTION SYNTHESIS WHEN SURVIVORS NOT VALUED IN PROPORTION TO KILL RATES

We now consider Case (b): $\frac{p}{q} > \frac{b_1}{b_2}$. Again, we work backwards from each possible type of end point of battle. We give the complete development for E_3: $x_1(T) > 0$, $x_2(T) > 0$, $y(T) > 0$, and $T = T_1$. There are two cases to be considered.

CASE (1): Never on singular subarc for finite interval of time.

Again there are two subcases to consider, depending upon whether the system winds up above or below L.

SUBCASE (1a): $a_1b_1x_1(t = T) \geq a_2b_2x_2(t = T)$

Since

$$v(\tau) = a_1b_1x_1\left(\frac{-p_2}{b_2}\right)\left[\frac{(p_1/p_2)}{(b_1/b_2)} - \frac{a_2b_2x_2}{a_1b_1x_1}\right],$$

we see that $v(\tau=0)>0$ and hence by (34) $\phi^*(t=T)=1$. Hence, for $0\leq\tau\leq\tau_1=$ the "backwards time" of the first switch, we use $\phi^*(\tau)=1$. We work backwards from the end using this control. Since

$$\frac{dv}{d\tau}=p_3(a_1b_1x_1-a_2b_2x_2)>0$$

when we are below L and we stay there by using $\phi^*(t)=1$, we have that $v(\tau)>0$ for all $\tau\epsilon[0,T]$, and hence we never switch. Thus, $\phi^*(\tau)=1$ for $0\leq t\leq T$.

SUBCASE (1b): $a_1b_1x_1(t=T)<a_2b_2x_2(t=T)$

Again there are two further subcases to consider, depending upon whether the system winds up above or below L'.

SUBCASE (1bI): $a_1b_1x_1(t=T)<a_2b_2x_2(t=T)$ and $a_1px_1(t=T)<a_2qx_2(t=T)$.

In this case we wind up above L' and hence by (39) $\phi^*(t=T)=0$. Since we are above L, $\frac{dv}{d\tau}<0$ for all τ by (40). Combining this with (38), it readily follows that $v(\tau)<0$ for all $\tau\epsilon[0,T]$. Thus, $\phi^*(t)=0$ for $0\leq t\leq T$.

SUBCASE (1bII): $a_1b_1x_1(t=T)<a_2b_2x_2(t=T)$ and $a_1px_1(t=T)>a_2qx_2(t=T)$.

In this case we wind up below L' at the end. By (38) and (39) we have that $v(\tau=0)>0$ and $\phi^*(\tau=0)=1$. We work backwards from the end. Since we are above L, $\frac{dv}{d\tau}<0$ by (40) while we remain above L. Thus $v(\tau)$ decreases as τ increases. There are two further subcases depending upon whether $v(\tau)$ decreases to zero before the line L is encountered. Let τ_1 be such that $v(\tau_1)=0$. If L has not been reached at τ_1, then $v(\tau)$ for $\tau>\tau_1$ is negative and $\phi^*(\tau)=0$ for $\tau_1\leq\tau\leq T$. It is also possible to just reach L when $v(\tau_1)=0$. In this case (assuming that we don't remain on the singular subarc) $v(\tau)>0$ for $\tau>\tau_1$, since we pass below L and then $\frac{dv}{d\tau}>0$.

CASE (2): on singular subarc for finite interval of time.

Considering (38) and (40), it is readily seen that this can only happen when

$$a_1b_1x_1(t=T)<a_2b_2x_2(t=T) \qquad \text{and} \qquad a_1px_1(t=T)>a_2qx_2(t=T).$$

As usual, we work backwards from the end of battle. By previous arguments it is read uly seen that we use $\phi^*(\tau)=1$ for $0\leq\tau\leq\tau_1$, and at $\tau=\tau_1$ we must have $a_1b_1x_1(\tau_1)=a_2b_2x_2(\tau_1)$. We use the singular control $\phi^*(\tau)=a_2/(a_1+a_2)$ for $\tau_1\leq\tau\leq\tau_2$. There are three further subcases.

(1) $\qquad x_1(\tau_2)=x_1^0, \qquad x_2(\tau_2)<x_2^0,$

(2) $\qquad x_1(\tau_2)<x_1^0, \qquad x_2(\tau_2)=x_2^0,$

(3) $\qquad x_1(\tau_2)=x_1^0, \qquad x_2(\tau_2)=x_2^0.$

We omit the trivial discussion of these cases.

Thus we see from the above that there are six possible cases for the extremal history of combatant force strengths in this prescribed duration battle:

(1) started below L and never reached L,

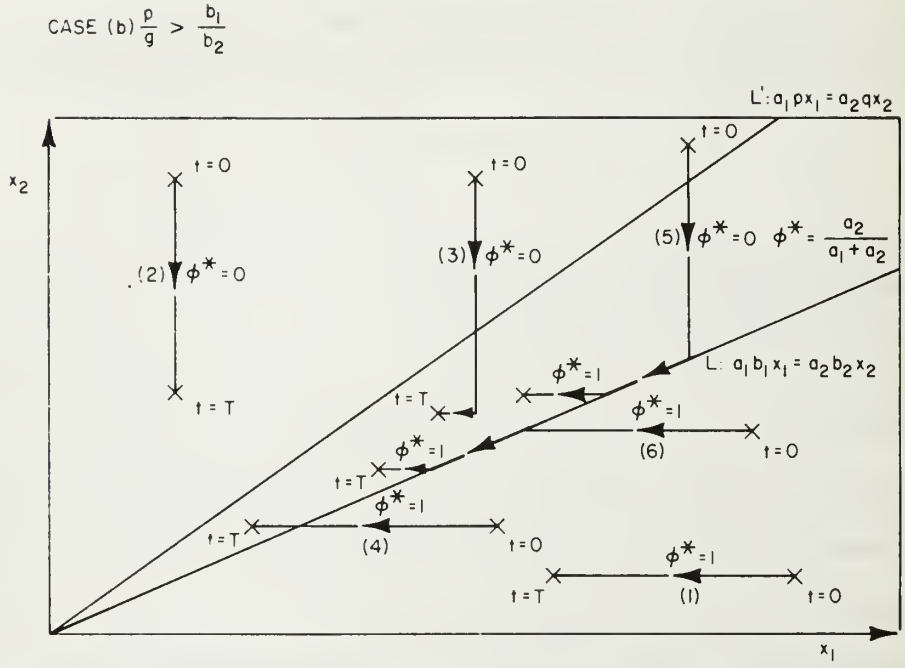


FIGURE 2. Optimal allocation for linear-law attrition process: survivors not valued in direct proportion to their attrition-rate coefficients

(2) always above L',

(3) started above L' and end up above L, but below L' without ever reaching L,

(4) end up above L, but started below L and did not remain on L for finite interval of time,

(5) started above (or on) L and were on L for finite interval of time,

(6) started below L and were on L for finite interval of time.

These six cases for extremals leading to E_3 are shown in Figure 2. Details are similar for extremals leading to E_2 and are, therefore, omitted.

It seems appropriate to make a few comments about extremals leading to E_1: $x_1(T) = x_2(T) = 0$, $y(T) > 0$, and $T \leq T_1$. We showed in section 5 that an optimal trajectory could only reach this terminal state (assuming that $x_1^0 > 0$ and $x_2^0 > 0$) by being on a singular subarc for $t_1 = T - \tau_1 \leq t \leq T$, and hence the dual variables have the boundary conditions shown in Table I. Thus, even in Case (b): $\frac{p}{q} > \frac{b_1}{b_2}$ an extremal leading to E_1 lies on the singular surface for $t_1 \leq t \leq T$. This situation should be contrasted with that for extremals leading to E_3 (such as (5) and (6) of Figure 2) as discussed above. In the case when extremals may lead to both E_1 and E_3 from a given region of initial force levels complete details have not been worked out (see section 7 for a further discussion). Except for this case, the extremals are unique and hence optimal.

Case (c): $\frac{p}{q} < \frac{b_1}{b_2}$ is similar to Case (b).

10. COMMENTS

Elsewhere [26] we have contrasted the structure of the optimal target engagement policies in Lanchester combat when the engaged target types undergo a "linear-law" attrition process with that for other tactical scenarios. An important question to be answered in such studies is whether target

priorities change over time. We have discovered that for the scenarios which we have so far studied the answer to this question is determined solely by whether or not surviving target types are valued in direct proportion to their kill-rate capabilities. For the case of constant attrition-rate coefficients, changes in target priorities over time can only occur when survivors are valued in excess of their kill-rate capabilities. This is true when the engaged target types are undergoing either a "linear-law" attrition process or a "square-law" one (see [27] for a discussion of the "square-law" case).

We now discuss how the above principle applies to the problem at hand. When a linear utility is assigned to enemy survivors at the end of battle in direct proportion to their kill-rate capabilities (as measured by their Lanchester attrition-rate coefficients) against friendly forces, then the optimal target selection policy depends only upon the location of the battle trajectory with respect to the singular "surface" L (see Figure 1). Thus, target priorities don't change over time (they can become equal, however). When one target type is assigned utility in excess of its effectiveness (i.e., $p/q > b_1/b_2$), then at time t_1 there will be a switch from tactics being determined by the location of the battle trajectory with respect to the singular "surface" L to being determined by location with respect to the line L' (see Figure 2). It may be shown that t_1 depends on the particular battle trajectory under consideration. Furthermore, no optimal trajectory can "penetrate" L'. (The proof of this statement is implicit in the details given in section 9. of our backwards construction technique for extremals. If an extremal terminates at $t=T$ above L', then $\phi^*(T)=0$ by (39) so that by working backwards from T we are led away from L'. A similar statement holds for an extremal terminating below L'. Hence, by construction an optimal trajectory cannot "penetrate" L'.)

The methodology for solution synthesis developed in this paper is applicable to more complex tactical situations of greater military significance. Our work here lays the foundations for the study of the optimal allocation of supporting weapon systems (e.g., artillery, tactical air support, etc.) against "area targets" (e.g., troop concentrations). Typical questions of interest to be answered are, "Considering several infantry companies individually engaging enemy units of like size, what is the 'best' utilization of supporting artillery fires?" or, "What is the 'best' utilization of Naval fire support in amphibious assaults?"

In a previous paper [26], we have pointed out that the structure of the optimal allocation policies in Lanchester combat is basically determined by whether there are constant attrition returns over time per unit of weapon system employed or diminishing returns. In the present paper we have studied target selection with diminishing returns over time, i.e., "linear-law" attrition process. It should be noted that there is a problem in the literature with similar solution structures, the continuous version of Bellman's stochastic goldmining process (see pp. 222–233 of [1]). When there are diminishing returns over time from the use of a device subject to breakdown, then the problem of maximizing the return from use of a device in either of two potential locations has a similar structure to the optimization problem in Lanchester combat studied here. The interested reader should compare the solution as shown in our Figure 1 with that of Theorem 1 on p. 231 of [1] and also our Figure 2 with Figure 4 on p. 323 of [1]. When the stochastic goldmining problem is reexamined by modern optimal control theory, new insights are gained into the operation of maximizing the return from a resource subject to breakdown or loss, and we shall discuss this in the future.

11. ACKNOWLEDGMENT

The author would like to thank the referee for his numerous suggestions for improving this paper. In particular, the referee suggested the discussions of the state variable inequality constraints and of the optimality of an extremal trajectory via citing an existence theorem for an optimal control.

Appendix

APPENDIX A. NONOPTIMALITY OF POLICY WHICH RESULTS IN $x_1(T)=0$ BUT $x_2(T)>0$

There are two cases to be considered (depending on whether or not we are on a constrained subarc for a finite interval of time):

(1) $x_1(T)=0$ with $x_1(t)>0$ for $T-\delta<t<T$ where $\delta>0$,
(2) $x_1(t)=0$ for $t_e \leqslant t \leqslant T$ $(t_e<T)$.

Each of these cases requires separate treatment.

CASE (1): $x_1(T)=0$ with $x_1(t)>0$ for $T-\delta<t<T$, where $\delta>0$.

There are two subcases to be considered: (1) $y(T)>0$, and (2) $y(T)=0$. In the first case, we have by (33)

$$v(\tau=0)=-qa_2x_2(T)<0, \tag{A.1}$$

and

$$\frac{dv}{d\tau}(\tau=0)=-ra_2b_2x_2(T)<0, \tag{A.2}$$

since $p_2(t=T)=p_2(\tau=0)=-q$ and $p_3(t=T)=r$. Considering a Taylor series expansion of $v(\tau)$ about $\tau=0$, one has that

$$v(\tau)<0 \qquad \text{for } 0 \leqslant \tau < \delta_1. \tag{A.3}$$

However, by (34) one sees that (A.3) implies that

$$\phi^*(t)=0 \qquad \text{for } T-\delta_1<t \leqslant T, \tag{A.4}$$

and hence it is impossible to have $x_1(T)=0$ but $x_1(t)>0$ for $t<T$.

In the second case in which $y(T)=0$, we have from (23) that $p_3(\tau=0)=0$. Then, one finds that

$$v(\tau=0)=-qa_2x_2(T)<0, \tag{A.5}$$

$$\frac{dv}{d\tau}(\tau=0)=0, \tag{A.6}$$

and

$$\frac{d^2v}{d\tau^2}(\tau=0)=-(a_2x_2(T))^2b_2q<0. \tag{A.7}$$

As above, one finds that $v(\tau)<0$ for $0 \leqslant \tau < \delta_2$, and this again leads to a contradiction. Hence, an optimal policy cannot result in Case (1).

CASE (2): $x_1(t)=0$ for $t_e \leq t \leq T$ $(t_e < T)$.

Again, there are two subcases to be considered: (1) $y(T) > 0$, and (2) $y(T)=0$. In the first case, we again observe that

(A.8) $$p_3(t=T)=r>0.$$

It is obvious that if we have destroyed X_1 (i.e., $x_1(t_e)=0$ for $t_e < T$), then the optimal policy must be to concentrate all fire on X_2. Thus

(A.9) $$\phi^*(t)=0 \qquad \text{for } t_e \leq t \leq T.$$

Following Bryson and Ho (see pp. 117–119 of [4]) the SVIC $x_1 \geq 0$ (or equivalently $-x_1 \leq 0$) may be transformed into a control variable inequality constraint by considering the point constraint

(A.10) $$x_1(t_e)=0,$$

where t_e denotes the entry time to the contrained subarc, and

(A.11) $$-\frac{dx_1}{dt}=a_1x_1y \leq 0 \qquad \text{for } t_e < t \leq T,$$

(see also chapter 6 of [22] and [14]). The constraint $0 \leq \phi \leq 1$ may also be written as $\phi(\phi-1) \leq 0$.

Then by the maximum principle [4] for (4) we are led to consider (at least formally) for $t_e < t \leq T$

(A.12) $$\underset{\phi}{\text{maximize}}\, H(t, x_i^*, p_i, \phi),$$

subject to:

$$-\frac{dx_1}{dt}=\phi a_1x_1y \leq 0,$$

$$\phi(\phi-1) \leq 0,$$

which by (7) is equivalent to

(A.13) $$\underset{\phi}{\text{maximize}}\, \phi(-p_1a_1x_1y+p_2a_2x_2y),$$

subject to:

$$\phi a_1x_1y \leq 0,$$

$$\phi(\phi-1) \leq 0.$$

On the constrained subarc with $x_1=0$, (A.13) reduces to

(A.14) $$\text{maximize } \phi p_2a_2x_2y,$$

subject to:

$$g(\phi) = \phi(\phi - 1) \leq 0,$$

for all $t \in (t_e, T]$. Considering the Kuhn-Tucker theorem, we form the Lagrangian function

$$\text{(A.15)} \qquad \mathcal{L}(\phi, \lambda) = \phi p_2 a_2 x_2 y - \lambda \phi(\phi - 1),$$

where

$$\lambda^* \begin{cases} = 0 & \text{for } \phi^*(\phi^* - 1) < 0, \\ \geq 0 & \text{for } \phi^*(\phi^* - 1) = 0. \end{cases}$$

According to the Kuhn-Tucker theorem, it is necessary for a maximum to (A.14) to occur at ϕ^* that

$$\text{(A.16)} \qquad \frac{\partial \mathcal{L}^*}{\partial \phi} = p_2 a_2 x_2 y - \lambda^*(2\phi^* - 1) = 0,$$

with $\lambda^* \geq 0$. Now $\phi^* = 0$ implies that

$$\text{(A.17)} \qquad \lambda^* = -p_2 a_2 x_2 y \geq 0.$$

(The existence of such a multiplier is guaranteed by the fact that $\frac{\partial g}{\partial \phi} \neq 0$ at $\phi^* = 0$.) Thus, we conclude that

$$\text{(A.18)} \qquad \phi^* = 0 \Leftrightarrow p_2(t) \leq 0 \qquad \text{for } t_1 < t \leq T,$$

the sufficient part of the assertion (A.18) holding by virtue of the fact that (A.14) is a concave programming problem and $x_2, y > 0$.

At the entrance to the constrained subarc at $t = t_e$, the following corner conditions hold [4], [14], [21]:

$$\text{(A.19)} \qquad p_i(t_e^-) = p_i(t_e^+) = p_i(t_e) \qquad \text{for } i = 2, 3,$$

and

$$\text{(A.20)} \qquad H^*(t_e^-) = H^*(t_e^+),$$

where t_e^- denotes a left-hand limit and $H^*(t_e^-)$ denotes $H(t_e^-, x_i^*, p_i, \phi^*)$. Now, we also have that

$$\text{(A.21)} \qquad H^*(t_e^-) = -p_2(t_e)\{1 - \phi^*(t_e^-)\} a_2 x_2 y - p_3(t_e) b_2 x_2,$$

where $\phi^*(t_e^-) > 0$ and

$$\text{(A.22)} \qquad H^*(t_e^-) = -p_2(t_e) a_2 x_2 y - p_3(t_e) b_2 x_2,$$

so that the corner condition (A.20) yields that

$$\text{(A.23)} \qquad p_2(t_e) \phi^*(t_e^-) a_2 x_2(t_e) y(t_e) = 0,$$

with $\phi^*(t_e^-) > 0$. Hence

(A.24) $$p_2(t_e) = 0.$$

On the constrained subarc where $x_1^* = 0$ for $t_e \leq t \leq T$, the adjoint equations read

(A.25) $$\frac{dp_2}{dt} = (1-\phi^*)a_2 y p_2 + b_2 p_3 \text{ with } p_2(t=T) = -q,$$

and

(A.26) $$\frac{dp_3}{dt} = a_2 x_2 p_2.$$

Now, (A.18) and (A.26) yield that

(A.27) $$p_3(t) \geq p_3(t=T) \qquad \text{for } t_1 \leq t \leq T.$$

In the first case considered above (i.e., with $y(T) > 0$), (A.8) and (A.27) yield that

(A.28) $$p_3(t) > 0 \qquad \text{for } t_1 \leq t \leq T.$$

Then (A.24) and (A.25) yield that

(A.29) $$\frac{dp_2}{dt}(t=t_e) = b_2 p_3(t_e) > 0.$$

However, if we were to have $x_1(t) = 0$ for $t_e \leq t \leq T$, then (A.24) and (A.29) would imply that $p_2(t) > 0$ for $t_e < t < t_e + \gamma$, which violates the necessary condition (A.18). Hence, when $y(T) > 0$ it is never optimal to have $x_1(T) = 0$, but $x_2(T) > 0$.

In the second case, T is unspecified and the transversality condition $H^*(t=T) = 0$ yields that when $y(T) = 0$ we have

(A.30) $$p_e(t=T) = 0.$$

Now since $p_2(t=T) = -q$, by (A.26) we may conclude that $p_3(t=t_e) > 0$, and again the reasoning following (A.28) above leads to a violation of (A.18).

APPENDIX B. NONOPTIMALITY OF POLICY WHICH RESULTS IN $x_1(t_2) = 0$ BUT $x_2(t_2) > 0$ WITH $t_2 < T$ AND $x_2(T) = 0$

Considering the principle of optimality [1] and the result of appendix A, it is clearly not an optimal policy to have $x_1(t_2) = 0$, but $x_2(t_2) > 0$ with $t_2 < T$ and $x_2(T) = 0$.

Let us now discuss the plausibility of the above. The result of these two appendices is intuitively obvious when one considers marginal returns per unit of weapon system allocated. As in [26] one considers

$$\frac{\left(-\frac{dx_1}{dt}\right)}{y} = a_1 x_1. \tag{B.1}$$

Hence, as x_1 is driven to zero, surviving units of x_1 are increasingly more difficult to destroy. Thus, as long as $x_2 > 0$, consideration of (B.1) shows the plausibility of the above result to the reader, since annihilation of x_1 with $x_2 > 0$ is accomplished under circumstances of vanishing marginal returns per unit of weapon system allocated.

REFERENCES

[1] Bellman, R., *Dynamic Programming* (Princeton University Press, Princeton, 1957).

[2] Berkovitz, L., "Necessary Conditions for Optimal Strategies in a Class of Differential Games and Control Problems," SIAM J. Control *5*, 1–24 (1967).

[3] Berkovitz, L. and S. Dreyfus, "A Dynamic Programming Approach to the Nonparametric Problem in the Calculus of Variations," J. Math. and Mech. *15*, 83–100 (1966).

[4] Bryson, A. and Y. C. Ho, *Applied Optimal Control* (Blaisdell Publishing Co., Waltham, Massachusetts, 1969).

[5] Cesari, L., "An Existence Theorem in Problems of Optimal Control, SIAM J. Control *3*, 7–22 (1965).

[6] Cesari, L., "Existence Theorems for Optimal Controls of the Mayer Type," SIAM J. Control *6*, 517–552 (1968).

[7] Davis, B. and D. J. Elzinga, "The Solution of an Optimal Control Problem in Financial Modelling," Opns. Res. *19*, 1419–1433 (1971).

[8] Funk, J. and E. Gilbert, "Some Sufficient Conditions for Optimality in Control Problems with State Space Constraints," SIAM J. Control *8*, 498–504 (1970).

[9] Isaacs, R., *Differential Games* (John Wiley & Sons, Inc., New York, 1965).

[10] Jacobson, D., "A New Necessary Condition of Optimality for Singular Control Problems," SIAM J. Control *7*, 578–595 (1969).

[11] Jacobson, D., "Sufficient Conditions for Nonnegativity of the Second Variation in Singular and Nonsingular Control Problems," SIAM J. Control *8*, 403–423 (1970).

[12] Jacobson, D., "A General Sufficiency Theorem for the Second Variation," J. Math. Anal. Appl. *34*, 578–589 (1971).

[13] Jacobson, D. and J. Speyer, "Necessary and Sufficient Conditions for Optimality for Singular Control Problems: A Limit Approach," J. Math. Anal. Appl. *34*, 239–266 (1971).

[14] Jacobson, D., M. Lele, and J. Speyer, "New Necessary Conditions of Optimality for Control Problems with State-Variable Inequality Constraints," J. Math. Anal. Appl. *35*, 255–284 (1971).

[15] Johnson, C., "Singular Solutions in Problems of Optimal Control," in *Advances in Control Systems*, C. Leondes (Ed.) (Academic Press, New York, 1965), Vol. 2, pp. 209–267.

[16] Johnson, C. and J. Gibson, "Singular Solutions in Problems of Optimal Control." IEEE Trans. on Automatic Control, Vol. AC–8, 4–15 (1963).

[17] Kelley, H., R. Kopp, and H. Moyer, "Singular Extremals," in *Topics in Optimization*, G. Leitman (Ed.) (Academic Press, New York, 1967), pp. 63–101.

[18] Lee, E and L. Markus, "Optimal Control for Nonlinear Processes," Archive for Rational Mechanics and Analysis *8*, 36–58 (1961).

[19] Lee, E. and L. Markus, *Foundations of Optimal Control Theory* (John Wiley & Sons, Inc., New York, 1967).

[20] Mangasarian, O., "Sufficient Conditions for the Optimal Control of Nonlinear Systems," SIAM J. Control *4*, 139–152 (1966).

[21] McIntyre, J. and B. Paiewonsky, "On Optimal Control with Bounded State Variables," in *Advances in Control Systems, Vol. 5*, C. Leondes (Ed.) (Academic Press, New York, 1967), Vol. 5, pp. 389–419.

[22] Pontryagin, L., V. Boltyanskii, R. Gamkrelidze, and E. Mishchenko, *The Mathematical Theory of Optimal Processes* (Interscience Publishers, Inc., New York, 1962).

[23] Speyer, J. and D. Jacobson, "Necessary and Sufficient Conditions for Optimality for Singular Control Problems; A Transformation Approach," J. Math. Anal. Appl. *33*, 163–187 (1971).

[24] Taylor, J., "Comments on a Multiplier Condition for Problems with State Variable Inequality Constraints," IEEE Trans. on Automatic Control, Vol. AC–17, 743–744 (1972).

[25] Taylor, J., "On the Isbell and Marlow Fire Programming Problem," Nav. Res. Log. Quart. *19*, 539–556 (1972).

[26] Taylor, J., "Lanchester-Type Models of Warfare and Optimal Control," Nav. Res. Log. Quart., to appear.

[27] Taylor, J., "Target Selection in Lanchester Combat: Heterogeneous Forces and Time-Dependent Attrition-Rate Coefficients," Nav. Res. Log. Quart., to appear.

AN *N*-STEP, 2-VARIABLE SEARCH ALGORITHM FOR THE COMPONENT PLACEMENT PROBLEM

Charles H. Heider

Center For Naval Analyses
Arlington, Virginia

ABSTRACT

The component placement problem is a specialization of the quadratic assignment problem that has been extensively studied for a decade and which is of considerable practical value. Recently, interest in component placement algorithms has risen primarily as a result of increased activity in the field of computer-aided design automation. This paper deals with the methodology of component placement and is based on the results of considerable operational experience. A tutorial presentation of tree search placement algorithms is provided, and an improved placement procedure is described which is demonstrated to be effective in generating near optimal solutions to the component placement problem. These solutions are completely reproducible and are obtained at an acceptable expenditure of computational resources. An additional objective is an assessment of performance of the class of near optimal algorithms. In particular, the question—how close to optimal are the near optimal solutions—is examined.

I. INTRODUCTION

The continuing trend toward microminiaturization in electronics with large scale integration, thin and thick film circuitry, and new wiring technology has produced radical changes in the way electronic systems are connected, interconnected and packaged. As miniaturization increased, component and interconnection density has become higher and higher so that the cost of designing and packaging a digital system now has become more expensive than the components themselves. The reduction of engineering design and packaging costs has therefore become an important objective in the goal to minimize total system costs.

Computer-aided design automation is thought by many to be the key to design cost reduction and much has already been accomplished toward this end. Two important steps in electronic systems design automation are component placement and interconnection routing. Performed sequentially, a satisfactory component placement contributes greatly to the ease with which a successful completion of the more difficult task of interconnection routing can be accomplished.*

This paper deals with the methodology of component placement. The history and relevant literature of the quadratic assignment problem and its variate the component placement problem is examined and a tutorial exposition of tree search placement algorithms is provided. An improved strategy for implementing the Graves and Whinston [6] implicit enumeration algorithm is introduced.

The Quadratic Assignment Problem has been investigated extensively by many researchers and a number of solution procedures have been developed.† Unfortunately, no computationally feasible algo-

*Refer to Reference [12] for a survey of computer-aided interconnection routing.

†Refer to Reference [14] for an extensive survey of the relevant literature and an excellent bibliography.

rithm has, as yet, been devised that will guarantee an optimal solution to a quadratic assignment problem except in exceedingly trivial cases. Algorithms have been developed, however, that are capable of generating good to near-optimal solutions to the QAP. The fact that the available QAP algorithms generate nonoptimal solutions is actually of little concern in many situations. The basis for this argument is strictly one of economics. A near-optimal solution obtained at a modest cost would generally be preferred over an optimal solution obtained at a much greater expense or over no solution at all.

A second objective of this paper therefore, is to assess the performance of the entire class of nonoptimal QAP algorithms. Using the Steinberg [15] test problem an experiment is conducted to ascertain just how near-optimal the solutions produced really are.

II. PROBLEM DESCRIPTION

The problem of interest can be described as follows: Given a set $S=(1, \ldots, m)$ of electronic circuit components and a circuit board with component location set $L=(1, \ldots, n)$ $n \geq m$. Determine the assignment of components to locations that will result in the minimum total length of interconnecting wire to electrically satisfy all required circuit connections. Stated in another manner, the problem is to determine the one-to-one mapping of the set S into the set L which will result in the minimum total wire length.

Associated with S is an n^2 interconnection matrix $F=|f_{ik}|$ $(i, k=1, \ldots, m)$ with $f_{ik} \geq 0$. Each f_{ik} representing the number of wires connecting component s_i with component s_k. Associated with L is an n^2 distance matrix $D=|d_{jq}|$ $(j, q=1, \ldots, n)$ with $d_{jq} \geq 0$. Each d_{jq} representing the distance between location j and location q. The length of the interconnecting wires is determined by forming the permuted dot product of F and D for a given mapping of S into L.

There are $n!$ unique one-to-one mappings of S into L, thus the feasible region of the problem's solution space contains $n!$ point. The assignment of components to locations can be recorded by means of an n^2 permutation matrix $X=|x_{ij}|$ $(i, j=1, \ldots, n)$ with $x_{ij}=0$ or 1. Alternately, the mapping can be recorded as a permutation of length n. Thus, $V_i=(v_1, v_2, \ldots, v_n)$ is a permutation of the integers $(1, 2, \ldots, n)$ or component numbers with the position in the permutation designating its assigned location.

Using the terminology from above, the component placement problem can be stated mathematically as follows: Given F and D, find X so as to

$$\text{(1)} \qquad \text{minimize } z=\sum_i \sum_j \sum_k \sum_q f_{ik} d_{jq} x_{ij} x_{kq}$$

$$(i, j, k, q=1, \ldots, n)^*,$$

subject to

$$\text{(2)} \qquad \sum_j x_{ij}=1 \qquad \text{for } (i=1, \ldots, n),$$

and

$$\text{(3)} \qquad \sum_i x_{ij}=1 \qquad \text{for } (j=1, \ldots, n)$$

*With no loss of generality it can be assumed that $m=n$. If not, then $n-m$ imaginary components may be added with associated $f_{ik}=0$.

also letting

$$x_{ij}=\begin{cases}1 & \text{if component } i \text{ is assigned to location } j\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

In some placement problems an initial fixed cost is often incorporated. This cost is represented by the n^2 matrix $C=|c_{ij}|$ $(i,j=1,\ \ldots,\ n)$ with $c_{ij}\geq 0$. Here c_{ij} is the cost of assigning component i to location j and is independent of any other component or location. This cost is exactly the cost considered in the linear assignment problem. If the fixed assignment cost is to be included then the objective function becomes:

$$\text{minimize } \sum_i \sum_j c_{ij}x_{ij}+\sum_i \sum_j \sum_k \sum_q f_{ik}d_{jq}x_{ij}x_{kq}. \tag{5}$$

The matrix C is frequently used to influence the final placement of the components. For example, if component i is constrained from being placed on location j then the coefficient c_{ij} would be set artificially high so as to discourage this possibility. Equation (5) along with Equations (2), (3), and (4) represent a more general formulation of the placement problem which has come to be known as the *quadratic assignment problem.* The linear term in Equation (5) is by itself, a linear assignment problem. The quadratic assignment problem is also closely related to the classical "traveling salesman" problem in which the shortest travel distance through a number of cities is desired. In fact, under certain conditions, the quadratic assignment problem reduces to the traveling salesman problem. These conditions are:

$$f_{ik}\begin{cases}1 & k=i+1,\ i<n\\ 1 & i=n,\ k=1\\ 0 & \text{otherwise.}\end{cases} \tag{6}$$

III. PLACEMENT ALGORITHMS: HISTORY AND RELEVANT LITERATURE

The problem of assigning indivisible entities to mutually exclusive locations has long been of interest to engineers, economists, and management scientists. Beginning in 1957, Koopmans and Beckmann [9], studying the problem of allocating plants to potential plant sites, formulated and identified this problem as the quadratic assignment problem. The combinatorial nature of the quadratic assignment problem makes the determination of the optimal solution difficult, if not impossible. Koopmans' and Beckmann's experience led to the conclusion that the computational difficulties associated with solving the quadratic assignment problem were insurmountable for problems of even moderate size (say $n=10$).

Complete or exhaustive enumeration is obviously impossible, as this approach would quickly become computationally infeasible as n increased. Research was, therefore, begun to search for an acceptable alternative to exhaustive enumeration. The literature of the quadratic assignment problem and its related variations is concerned with the possible alternatives that have been conceived and investigated.

Since 1957, Gilmore [5] and Lawler [10] reported optimal algorithms which were computationally feasible only for very small problems. Gilmore reported that optimal algorithms are probably not computationally feasible for n much larger than 15.

In 1968, the team of Nugent, Vollman, and Ruml [13] concluded, as a result of their investigations, that the probability of obtaining a computationally feasible optimal algorithm was very remote. Instead they suggested that interest must be focused on the development of suboptimal procedures.

Most research has, in fact, been focused on the search for an acceptable suboptimal algorithm which would be computationally feasible for large problems and still produce acceptable solutions.

In 1961, Steinberg [15] reported the development of a suboptimal algorithm for a similar problem which he identified as the backboard wiring problem.* Steinberg tested his algorithm on a problem requiring the assignment of 34 components to a possible 36 locations on a (4×9) backboard. This problem has subsequently become known as the Steinberg test problem and now serves as a bench mark test for the comparison of new algorithms. Improved suboptimal algorithms have also been reported by Gilmore [5] in 1962, by Hillier and Conners [8] in 1966, and also by Graves and Whinston [6] in 1966. Each of these algorithms in turn was able to better the previous best solution when tested on the Steinberg problem. A much more recent contribution has been reported by Gaschutz and Ahrens [3] in 1968 which has held the record for the minimum solution value to the Steinberg test problem. It should be noted, however, that this algorithm requires excessive computation time and produces solutions that are not reproducible on successive applications. Thus, considering computational effort and reproducibility, the algorithm of Graves and Whinston [6] must currently be considered to be the most acceptable to date.

Almost every conceivable approach has seemingly been investigated in the search to produce a good placement algorithm. The more successful algorithms to date have used an implicit enumeration scheme known as "branch and bound." The work of Gilmore [5], and Hillier and Conners [8], and Graves and Whinston [6] have all relied on some variation of "branch and bound" techniques.

Other approaches have been the heuristic algorithms of Armour and Buffa [1] and Hillier [7]. Breuer [2] and Lawler [10] investigated the possibility of reformulation as an integer program with the solution then being determined by integer programming techniques. Steinberg's [15] algorithm employs a graph theoretic approach. Reiter and Sherman [16] suggested a probabilistic search procedure for a general class of discrete optimization problems which they then applied to the traveling salesman problem. Nugent et al. [13] investigated a similar sampling procedure which they referred to as biased sampling. Finally, Gaschutz and Ahrens [3] produced a multistep procedure combining various approaches including graph theoretic, linear programming methods, and sampling procedures.

Of all the approaches described, application experience supports the conclusion that "branch and bound" based algorithms currently produce the most acceptable solutions when judged on the combined criteria of solution value, reproducibility, and execution time. Reproducibility and execution time can be equated in terms of dollars to the cost of attaining a particular solution value with a given algorithm. Hence, when the expected cost of obtaining a certain solution with a specific algorithm exceeds the expected return or value of this solution, continued use of this algorithm is no longer justifiable.

IV. SEARCH-TREE ALGORITHMS

The more common name applied to the controlled enumeration procedures to be described is "branch and bound," the name given to the ideas employed by Little et al. [11] in their algorithm for

*As noted previously, the backboard wiring problem along with the component placement problem are variations of the quadratic assignment problem where $C = \emptyset$ the null matrix.

the traveling salesman problem. "Branching" refers to the fact that in terms of a tree of alternate potential solutions to the problem, the procedure is continuously concerned with choosing the next branch of the tree to elaborate and evaluate. The "bound" term denotes the emphasis on, and effective use of, means for bounding the value of the objective function at each note for eliminating dominated paths. "Branch and bound" procedures are based on two concepts: the use of a controlled enumeration technique for implicitly considering all feasible solutions; and the elimination from consideration of particular solutions which are known from dominance, bounding, and feasibility considerations to be unacceptable.

The Search-tree and the Branching Process

Originally, the basis of "branch and bound" was a two-dimensional search-tree which is constructed as illustrated in Figure 1. The vertical dimension of the tree represents the order or level of the search and has associated with it, one of the n locations. The pairings of the locations to levels are accomplished a priori to the application of the search procedure and remain fixed throughout the tree search. The order in which the locations are fixed to the tree levels is critical as different pairings will produce different permutations V_g and final values $Z(V_g)$. Gilmore suggests the following "rule of thumb" for choosing an appropriate pairing set from the set of possible pairings. Fix the locations to levels based on the decreasing order of

$$\sum_{j,q} (d_{jq}+d_{qj}) \qquad (j, q=1, \ldots, n).$$

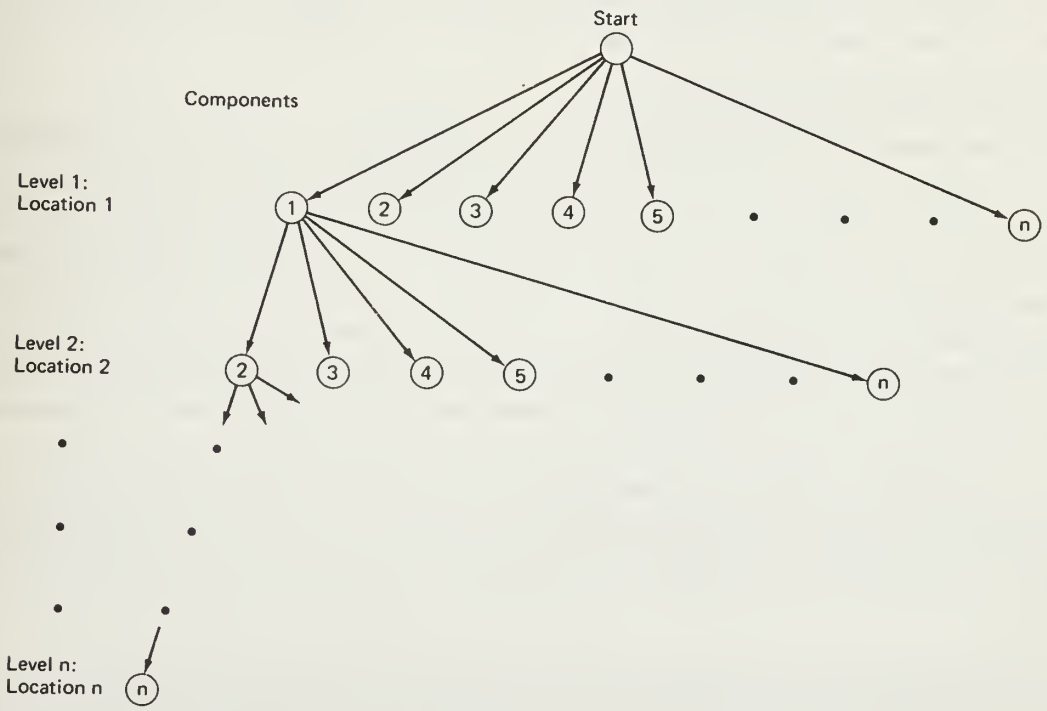


FIGURE 1. Two-dimensional search tree—single variable branch and bound

The *free* (decision) variable in the search procedure is the component to be assigned at the respective levels. At level 1, the search considers the potential assignment of each of n components to the location that has been assigned to level 1. At level 2, $(n-1)$ components are considered for possible assignment to the location associated at level 2. A permutation $V=(v_1, v_2, \ldots, v_k, \ldots, v_n)$ specifies the assignment of the n components to the n locations, such that v_k is the number of the component assigned to location j, at level k, $j=k$.* At any level k, $(k-1)$ components would have been assigned, leaving $(n-k+1)$ unassigned components to an equal number of locations.

There are a possible $n!$ permutations with an equal number of tree search paths and feasible solutions. At least one of the paths in the tree represents a minimum solution to the problem. The determination of the minimum valued path is normally the objective of any search procedure. The procedure requires $(n-1)$ steps to completely elaborate the tree since after the $(n-1)$ step there will remain only a single unassigned component. This results in a default placement to the remaining unassigned location.

The two-dimensional search-tree has provided one of the more successful approaches reported to date for constructing acceptable placement algorithms. Yet several drawbacks are evident which diminish the value of this approach. They are:

1. The necessity of having to order and preassign the location to levels prior to the algorithm application.
2. The lack of reproducibility among users due to the use of different location to level assignments.
3. The probability of obtaining an inferior solution due to the order in which the locations are considered.

All three defects are essentially related to the preassignment problem. Thus, the elimination of this requirement should potentially improve the performance of this method.

Two-variable Branch and Bound

The next logical step in the development of tree search algorithms is the three-dimensional search-tree. The addition of a second variable to the enumeration procedure extends the search-tree into three-dimensional space as illustrated in Figure 2.† Both the component and location are treated as free variables which results in n^2 decision possibilities at level 1 versus n decision possibilities in the single-variable case. The three-dimensional search-tree can be thought of as an inverted pyramid with the pyramid base containing n^2 nodes.

As was the case in the single-variable algorithm, this procedure also begins at level 1 in the search-tree and proceeds sequentially through $n-1$ iterative steps. The difference is that an assignment pair (location, component) is now chosen at each level.

The advantage of the three-dimensional search procedure is that the faults attributed to the two-dimensional search are eliminated. On the other hand, the computational requirements are increased. This apparent disadvantage can be minimized, however, in constructing an efficient bounding process.

*This assumes that the location have been numbered to correspond to the level number to which it has been assigned.

†The construction of an $N \times M$ search-tree was initially proposed by Graves and Whinston [6], however, it will be demonstrated that a more efficient procedure results from the construction of an $N \times N$ tree where $(N-M)$ imaginary components are added to the component set. The tree then has N levels as opposed to Graves' and Whinston's M level tree.

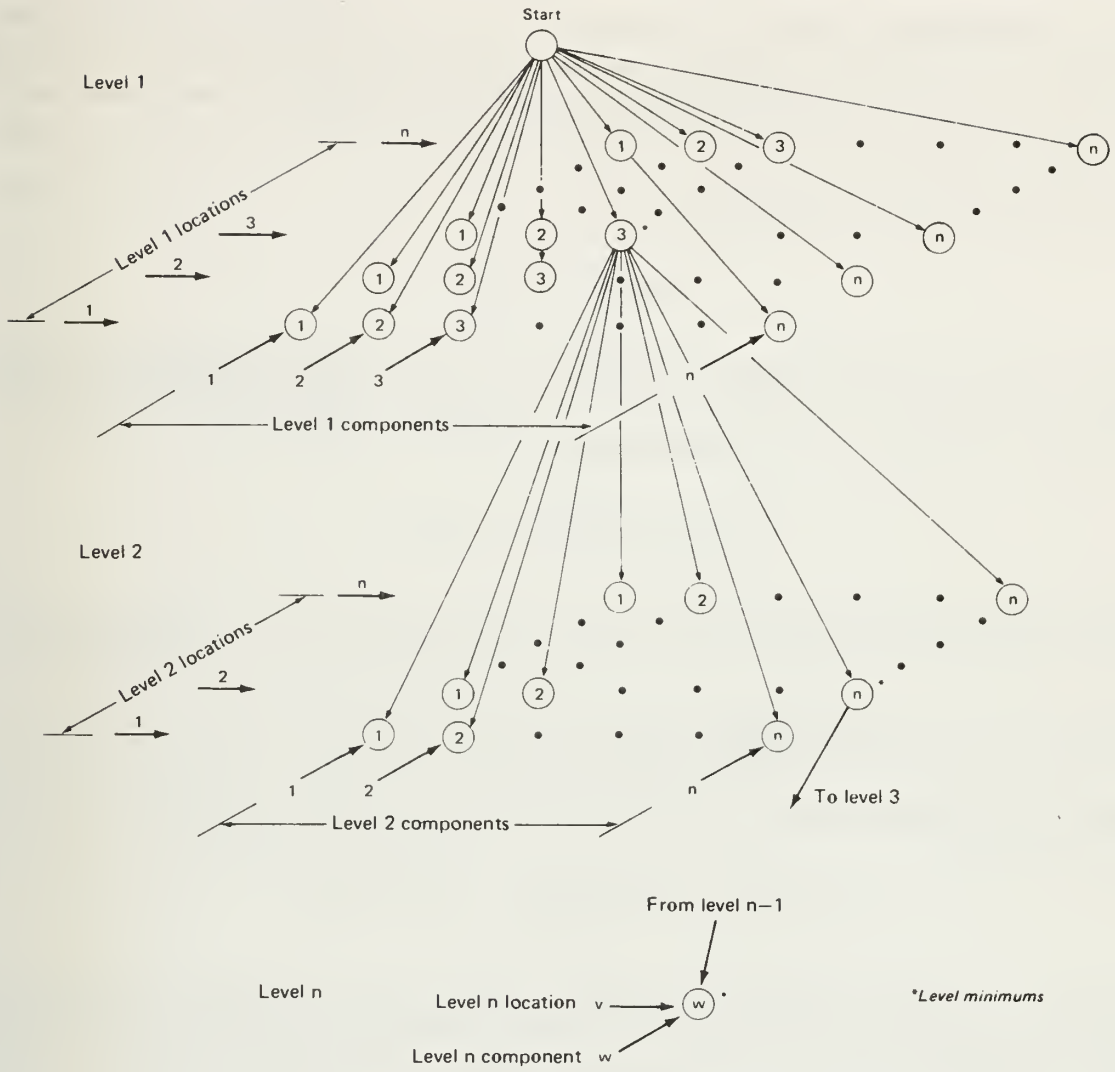


FIGURE 2. Three-dimensional search tree

The Bounding Process

The method of determining the bounding or dominance measure for implicitly eliminating solutions from further consideration is the area where tree search algorithms differ. The algorithms of Gilmore [5], and Graves and Whinston [6] differ significantly in this respect. Before discussing these algorithms specifically, however, the general process of bounding will be discussed.†

First, considering the two-dimensional search procedure, let V_g designate permutation g, and $Z(v_g)$ the criterion value of V_g. Then for any other permutation V_h, V_g dominates V_h if $Z(v_g) \leq Z(v_h)$. Thus, in the bounding process an attempt is made to rule out of further consideration permutations which are so dominated by another. During the enumeration procedure, the permutation V_g is only

†The use of the linear assignment problem as a bounding process, as was suggested by Lawler, has been excluded. This is done because of the computational problems (mainly execution time requirements) necessary in setting up and solving a large number of linear assignment problems.

partially complete (designated by $\bar{V}_g$) since at level k, $\bar{V}_g = (v_1, v_2, \ldots, v_k, 0_{k+1}, \ldots, 0_n)$. Because $Z(V_g)$ cannot be computed with partial $\bar{V}_g$, an estimate of $Z(V_g)$ is therefore introduced as a substitute in the dominance test. At level $k+1$ the bound B_j^{k+1} is computed on the potential placement of candidate component j from amongst the $n-k$ unassigned components ($j=1, 2, \ldots, n-k$) to location $k+1$ as an estimate of $Z(V_g)$ for partial $\bar{V}_g$.

The bound B_j^{k+1} consists of three parts, that is

$$B_j^{k+1} = b_1 + b_2 + b_3.$$

Letting S again be the set of unassigned components, S' the set of assigned components, L the set of unassigned locations and L' the set of assigned locations, then B_j^{k+1} is computed as follows. First, for the k assigned components in S', including candidate j, the sum b_1 is found through

$$b_1 = \sum_i \sum_j f_{ij} d_{1(i)1(j)}^* \qquad (i, j \epsilon S').$$

Second, the interactive contribution between members of S and members of S' is estimated from

$$b_2 = 2 \sum_i \sum_j G(f_{ij}, d_{1(i)q}) \qquad (i \epsilon S', j \epsilon S, \text{ and } q \epsilon L).$$

Finally, an estimate of the contribution from the unassigned components is computed

$$b_3 = \sum_i \sum_j \sum_k \sum_q G(f_{ij}, d_{kq}) \qquad (i, j \epsilon S, k, q \epsilon L).$$

The available tree search algorithms differ primarily in the operator G used in the determination of the bounds components b_2 and b_3. The bound B_j^{k+1} is computed for each candidate j ($j=1, \ldots, n-k$) with the candidate component being selected which produces the minimum value of B_j^{k+1}. The permutation V_g (actually $V_g = S'$) is then augmented with this component and becomes $V_g = (v_1, \ldots, v_k, v_{k+1}, 0_{k+2}, \ldots, 0_n)$. The selection process then advances to level $k+2$ and repeats the bounding process. The complete enumeration of the two-dimensional search-tree requires the computation of T bounds in total where

$$T = \sum_i (n+1-i) \qquad (i=1, \ldots, n).$$

For the improved three-dimensional search procedure, the selection of component and location is again based on the computation of a lower bound, but the bound is now B_{ij}^k signifying a bound with component i fixed to location j at level k. The minimum B_{ij}^k is again selected with the minimum producing pair (i, j) being made the permanent assignment at level k. The increased dimension of the search-

*Read $1(i)$ as the location of i.

tree requires the computation of

$$\sum_{k=1}^{n} (n-k+1)^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

bounds, compared to

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

for the single-variable algorithm.

Thus, the 2-variable algorithm requires

$$\frac{2n+1}{3}$$

times more computations than the single-variable procedure. There are three bounding processes available. They are:

(i) Minimum permuted dot product.

The operator G employed by Gilmore [5] consists of the minimum permuted dot product formed on the proper terms of F and D. This is accomplished as follows: Given two vectors $U = (u_1, u_2, \ldots, u_m)$ and $W = (w_1, w_2, \ldots, w_m)$ of nonnegative elements, the problem of determining a permutation r of $(1, \ldots, m)$ for which the dot product

$$\sum (u_1 w_{r(i)})$$

is a minimum is not difficult. It is only necessary to match the smallest u_i and largest w_j, the second smallest u_i and second largest w_j and so forth. For any two vectors U and W of equal dimension, the minimum permuted dot product will be represented as $P(u, w)$. Then the Gilmore bounds are computed as follows.*

$$B = \sum_{i,j} f_{ij} d_{1(i)1(j)} \qquad (i, j \epsilon S')$$

$$+2 \sum_{i} \sum_{j} P(f_{ij}, d_{1(i)q}) \qquad (i \epsilon S', j \epsilon S, \text{ and } q \epsilon L)$$

$$+ \sum_{i,j,k,q} P(f_{ij}, d_{kq}) \qquad (i, j \epsilon S, k, q \epsilon L).$$

In converging toward a feasible solution in successive iterations (iterations $= n$), the procedure of Gilmore begins with the minimum possible (most likely infeasible) product of all elements of F and D and attempts to maintain this value by allowing only the smallest increment to be added as a result of making an assignment.

*In Gilmore's n^4 algorithm.

(ii) Value of the associated mean.

Graves and Whinston [6] used the value of the associated mean as the basis of their bounding process which they refer to as a completion class evaluator (or CCE) rather than a bound. It has the advantage or requiring very few multiplication operations and therefore can be computed with much less effort than the Gilmore bounds.

Letting S, S', L and L' be as previously defined, then the Graves and Whinston bound is computed as follows:

$$B=\sum_{i,j} f_{ij}d_{1(i)1(j)} \qquad (i, j\epsilon S')$$

$$+\frac{2}{n-k}\sum_{i}\left(\sum_{j} f_{ij}\right)\left(\sum_{k} d_{1(i)k}\right) \qquad (i\epsilon S', j\epsilon S, k\epsilon L)$$

$$+\frac{1}{(n-k)(n-k-1)}\left(\sum_{i,j} f_{ij}\right)\left(\sum_{k,q} d_{kq}\right) \qquad (i, j\epsilon S, k, q\epsilon L).$$

The Graves and Whinston algorithm begins with the expected value of all feasible solutions and proceeds to attempt to decrement this value maximally at each level in the search-tree until converging to the final suboptimal solution value. An important feature of this completion class evaluator is that it is easily modified to solve the quadratic assignment problem with objective function as defined by Equation 5. In this situation, two additional components are added to the bound B which then becomes

$$B=b_1+b_2+b_3+b_4+b_5,$$

where

$$b_4=\sum_{i} c_{i1(i)} \qquad (i\epsilon S')$$

$$b_5=\frac{1}{n-k}\sum_{i}\left(\sum_{j} c_{ij}\right) \qquad (i\epsilon S, j\epsilon L).$$

The c_{ij} are the coefficients of the imbedded linear assignment problem.

(iii) Maximum permuted dot product.

A third bounding process not previously mentioned in the literature is suggested as a logical extension of the previously described procedures. This is the formation of the maximum permuted dot product in the reverse manner of forming the minimum. That is, by multiplying the largest element of F with the largest element of D, the second largest member of F with the second largest from D and so forth. This approach would have the effect of beginning with the worst possible value as the lower bound and then proceeding to select the alternative at each level that minimizes the worst that can possibly occur at each level. The value of the maximum permuted dot product when compared

to the other two procedures is not immediately obvious except that the rate of change in the minimum lower bound at each level is much greater. This aspect of the bounding processes will be discussed in the section on convergence.

Three procedures for computing bounds have been presented with no indication of relative value being made. Any ranking of the three methods must, however, be based in part on execution time requirements as well as the value of the solution produced. When judged by these criteria, the mean process is far superior with respect to execution time requirements. Only four multiplications are required in the computation of each (CCE) whereas $(n^2-n)/2$ multiplications are required by the other two methods in addition to the requirement for extensive sorting of vector elements. Both multiplying and sorting requires much more execution time than the simple additions and subtractions that are used in the computations of the value of associated mean bound. Thus, on a computational basis, the value of the associated mean process is far more efficient than either permuted dot product bounding process. Additionally, the mean value process has the advantage of being applicable in solving the more general quadratic assignment problem.

In view of this fact, the value of the associated mean bounding process is the more efficient choice to use in conjunction with the two-variable enumeration procedure because of its considerably lower execution time requirement.

Detailed Enumeration Procedure

The detailed enumeration procedure for the two-variable algorithm is as follows:

STEP 1. Beginning at level 1, initialize the unassigned component vector $S=(1, 2, 3, \ldots, n)$ and the unassigned location vector $L=(1, 2, 3, \ldots, n)$. Also initialize the assigned component vector $S'=(0, 0, 0, \ldots, 0)$ and the assigned location vector $L'=(0, 0, 0, \ldots, 0)$. Let B^* record the current minimum lower bound B_{ij}^k at level k for candidate component i and candidate location j. Initially set $B^*=2^{30}$, an arbitrary large number. Let i be an indicator that points to the candidate component of S that is currently being considered for permanent transfer to S'. Initially, let $i=0$. Let j be an indicator that points to the candidate location of L that is currently being considered for permanent transfer to L'. Initially, let $j=0$. Let k be the index that records the current level of the search. Initially, let $k=1$. Let n' designate the lengths of S' and L' and set $n'=0$. Let nn record the initial problem size and set $nn=n$, where n is the initial length of S and L.

STEP 2. Increment $i=i+1$. Then remove the ith component from S and assign it temporarily to the kth location of S'. The length of the vector S is now reduced from n to $n-1$, that is, $n=n-1$. The identification of the ith component is recorded by a marker t. *Set* $j=0$.

STEP 3. Increment $j=j+1$. Then remove the jth location from L and assign it temporarily to the kth location of L'. The identity of the jth location is recorded by a marker tl.

STEP 4. Now calculate the bound B_{ij}^k by means of a bounding process, preferably the value of the associated mean process.

STEP 5. Compare B_{ij}^k with B^*. If $B_{ij}^k \leq B^*$, let $B^*=B_{ij}^k$. Record the minimum bound producing component i by i^* and location j by j^*.

STEP 6. Replace the jth location marked by tl back into L.

STEP 7. Test to check if all candidate locations have been evaluated, that is, if $j+1>n$. If not, go back to step 3.

STEP 8. Replace the ith location marked by t back into S.

STEP 9. Test to check if all candidate components have been evaluated, that is, if $i+1>n$. If not, go back to step 2.

STEP 10. After all candidate components in S and locations in L have been examined for possible transfer to S' and L', select the minimum bound producing pair identified by the indicators (i^*, j^*). Permanently assign i^* to S' and j^* to L', both to the kth position.

STEP 11. Test to check if all nn levels of the search tree have been elaborated, that is, if $k+1>nn$. If not, set $i=0$, $B^*=2^{30}$ and return to step 2.

STEP 12. *Stop!* The enumeration process is finished and S' and L' now contain the permanent assignments of components and locations. Compute $Z(S'(L'))$ to obtain the objective function value.

Convergence

There are essentially three initial bounds which can be established on the optimal solution Z^*. These bounds are easily computed from available information contained in the F and D matrices and are:

1. *Min* B^0—the absolute minimum lower bound on the optimal solution Z^*, most likely an infeasible solution, which is computed from $\min P(f, d)$ with $S'=\phi$.

2. *Max* B^0—the absolute maximum upper bound on the optimal solution Z^*, most likely an infeasible solution, which is computed from $\max P(f, d)$ with $S'=\phi$.

3. *Mean* B^0—the expected value of all solutions, computed from

$$\left(\sum_{i,k}(f_{ik})\cdot\sum_{j,q}(d_{jq})\right)/n(n-1).$$

The bounds $\min B^0$ and $\max B^0$ establish the range of all solutions (feasible and infeasible) while the bound mean B^0 separates the set of all solutions into two subsets. In all probability, Z^* will belong to the lower valued subset and will lie much closer to $\min B^0$ than to mean B^0, such that

$$\min B^0 \leqslant Z^* \leqslant\leqslant \text{mean } B^0 \leqslant\leqslant\leqslant \max B^0.$$

Convergence refers to the process through which the current active bound B_{ij}^k approaches the final feasible solution value $Z(S')$ in the branch and bound procedure. Three convergence paths are possible for the n-step procedure: First, by use of the $\min B^0$ as the initial bound on the optimal solution, the branch and bound procedure can be executed using the minimum permuted dot product bounding process. At each of the n-search tree levels, the lowest bound B_{ij}^k is selected as the new bound on the optimal solution. Finally, at level n, $B_{ij}^n=Z(S')$, the final feasible solution value.

Unfortunately, there is no guarantee that the bound B_{ij}^k will not exceed Z^* at some level k, prior to the completion of the n-step branching process. Therefore, no statement regarding optimality can be made concerning the application of the branch and bound procedures to the nonlinear assignment problem. This statement also holds for the use of backtracking. Currently, the optimality of the final solution can be established only by the following:

1. If, by chance, $Z(S')=\min B^0$, then $Z(S')=Z^*$ and is the optimal solution.

2. Otherwise, all $n!$ feasible solutions must be exhaustively enumerated to determine Z^* and to verify optimality.

The initial rate of change in the bound B_{ij}^k will be relatively small compared with its subsequent rate since the lower bounding rule makes the component selection at each level that produces the minimal incremental change in the current lower bound on Z^*. However, as more and more placements are fixed, the number of choices at each succeeding level is reduced and the incremental difference in succeeding bounds grows larger. This effect is illustrated in Figure 3, by curve *A*.

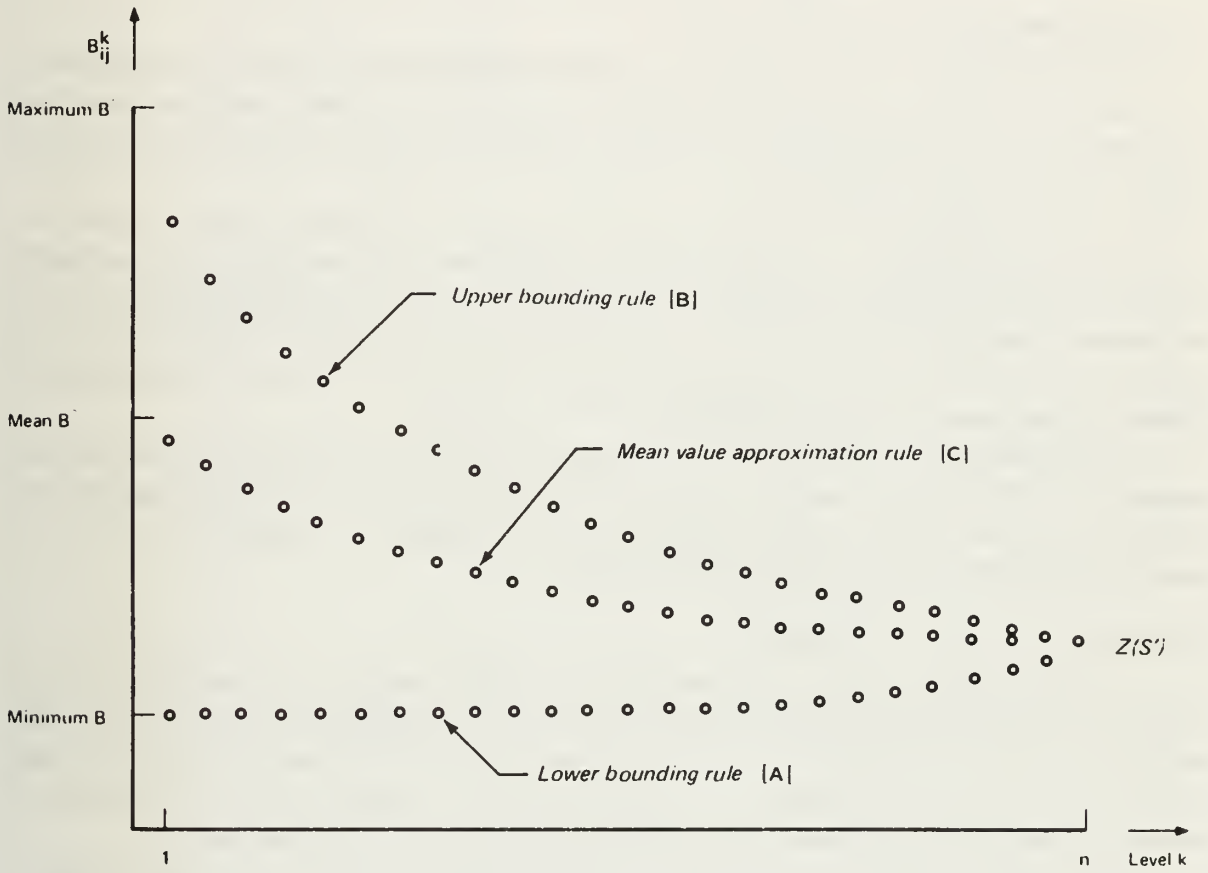


FIGURE 3. Convergence of branch and bound algorithms

The second bounding rule begins with the bound max B^0 and uses the maximum permuted dot product bounding process. Here, the value max B^0 is the absolute upper bound on the solution Z^*. During the n-step search procedure, the bound B_{ij}^k that produces the maximal decrement in the current upper bound on Z^* is selected at each level. Thus, the initial slope of this curve is much greater than the final slope which results when fewer choices become available. The convergence curve for the upper bounding rule process is shown in Figure 2, curve *B*.

A third bounding rule utilizes the value of the associated mean bounding process and begins with mean B^0 as the initial bound on the optimal solution Z^*. This process is essentially an approximation of the upper bounding rule where the mean B_i^k is substituted for max B_{ij}^k because its determination requires considerably less computational effort. The third rule is, therefore, also an upper bounding rule. Upper bounding rules cannot be used effectively when backtracking is to be employed because domi-

nance cannot be established until the final level in each path. The upper bounding rule is effective, however, for the n-step branch and bound procedures that are of interest here for solving nonlinear assignment problems. An example of the convergence curve for the value of the associated mean bounding process is shown in Figure 3, curve C. In this example, the slope will be less than for the curve B due to the initial starting value and because of the averaging effect of the bounding process.

V. EXPERIMENTAL RESULTS

This section describes a series of experimental tests which were conducted for the purpose of illustrating the application and the relative merits of the n-step, two-variable search algorithm. Two test problems were analyzed; first, the simple ($n=4$) problem of Gavett and Plyter [4] and second the larger ($n=36$) problem of Steinberg [15].

The first example is presented to demonstrate the nature of the placement problem and the mechanics of the related solution methodologies. Since the solution of this problem is trivial, the results obtained *should not* be used as the basis for comparing the performance of the various algorithms. For the purpose of comparison, the more complex Steinberg problem is presented. The minimum solution value for this problem is not known exactly. However, an attempt has been made to characterize the solution space by means of sampling procedures.

The computations in the test problem analyses were performed on a CDC 3800 with the programs written in FORTRAN. Initial work was performed on an IBM 360/75. For comparison purposes, the CDC 3800 is approximately 4–5 times slower than the IBM 360/75 and roughly 20–30 times slower than the IBM 360/91 used in Graves and Whinston [6].

The Gavett and Plyter Test Problem

The sample problem which will be described was initially used by Gavett and Plyter in their research on the optimal assignment of facilities to locations.* This problem is concerned with the optimal allocation of four facilities to four possible locations. Each of the facilities is required to transfer material between the other facilities to the extent shown in the diagram of Figure 4. The interplant material transfers can be represented by the traffic intensity matrix† F as shown in Figure 5.

The four facility locations along with their respective interlocation distances are shown in Figure 6. Again, these values are converted to matrix form as shown in Figure 7. The objective in this problem is, of course, to assign the facilities to locations so that the sum of the products of material quantities transferred by distance traveled is minimized.

A two-dimensional search-tree was initially constructed to demonstrate the combinatorial aspects of the placement problem. This tree, shown in Figure 8, contains 64 nodes and 24 ($n!$) paths with an equal number of feasible solutions. The solution value associated with each path was determined through exhaustive enumeration. It can be observed in the search-tree that the optimal (minimum) solution is 406 for the assignment (2, 4, 3, 1). The range of feasible solutions is quite broad—406 to 607.

---

*The problem analyzed varies slightly from the Gavett and Plyter problem in that $f_{31}=8$ rather than 5. See Figure 5. This error was initially made by Pierce and Crowston in Reference [13], but corrected in their symmetric matrix $f_{ik}+f_{ki}$. Thus, the value produced in this study will differ slightly.

†It should be noted that the traffic intensity matrix of the facility location problem corresponds to the interconnection matrix in the component placement problem formulation.

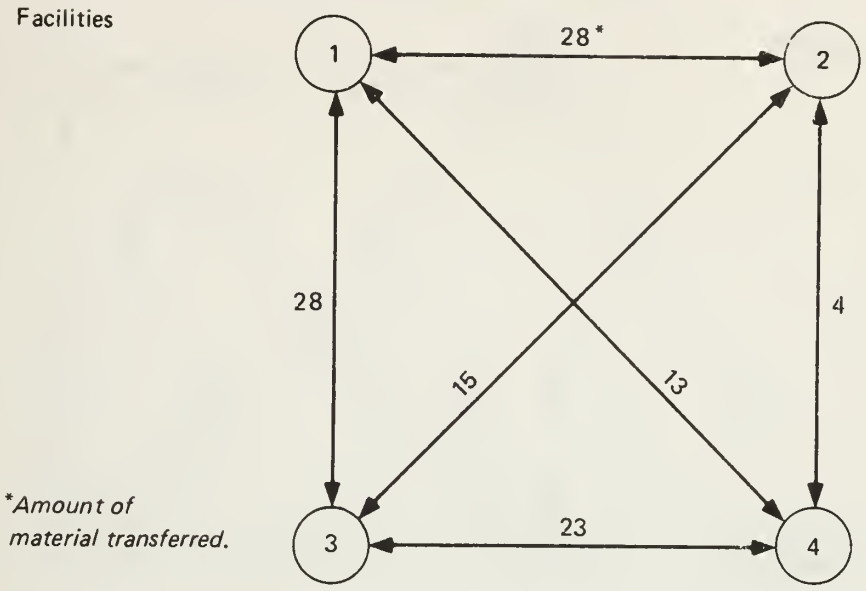


FIGURE 4. Traffic intensity diagram

Component

Component	1	2	3	4
1	–	10	20	5
2	18	–	9	4
3	8	6	–	8
4	8	0	15	--

FIGURE 5. Traffic intensity matrix F

The problem was solved once with each of the six possible procedures using tree search techniques. First the two-variable algorithm was used with each of the three bounding procedures. The numerical values of the bounds computed at each level for each of the three trials are recorded in Tables 1, 2, and 3.

It can be observed that in each case the optimal solution was obtained, but that the order of the location assignments was quite different. For example, the 2-VBB algorithm with the min permuted dot product bounding process assigned the locations in the order (1, 4, 2, 3) as opposed to the Gilmore "rule of thumb" which suggested the order (2, 1, 3, 4).

The three-dimensional search-tree is constructed by placing the 16 values of the "level 1" column from Table 1 into the appropriate nodes at level 1 of the search-tree as illustrated in Figure 9. Thus,

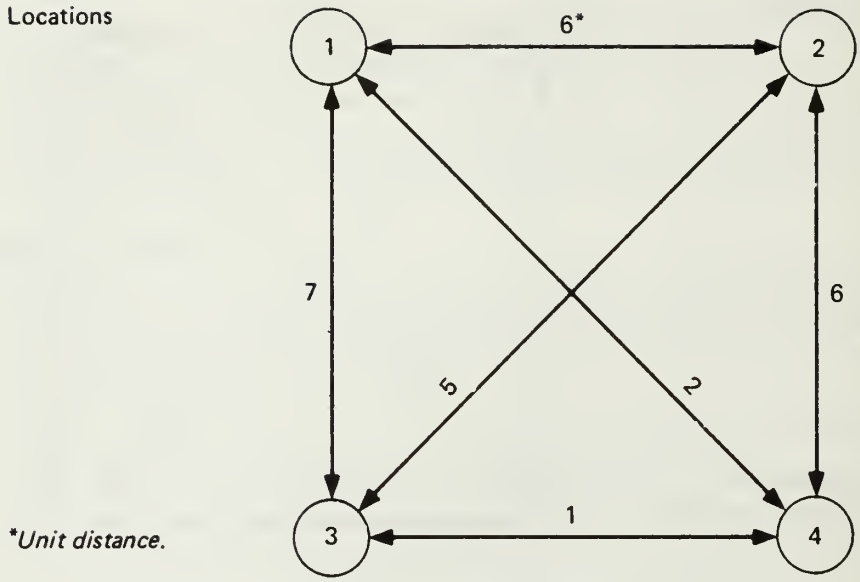


FIGURE 6. Location distance diagram

Location

Location	1	2	3	4
1	–	6	7	2
2	6	–	5	6
3	7	5	–	1
4	2	6	1	–

FIGURE 7. The distance matrix D

it can be observed in Figure 9 that two choices existed at level 1 due to a tie for the minimum lower bound. Either facility 1 to location 4 or facility 2 to location 1 could have been selected for permanent assignment at level 1 based on the minimum bound value of 395. In this example, the second selection was made. At level 2 there were then nine choices left corresponding to the nine nodes in the search-tree at level 2. The minimum bound of 395 occurred at level 2 for the assignment pair (1, 4). At level 3 only four choices remained with the pair (4, 2) being selected on the basis of the bound 401. Finally at level 4, the choice was made by default since only one possibility remained. The same general procedure applies to Tables 2 and 3, except that the numerical values of the lower bounds are considerably different.

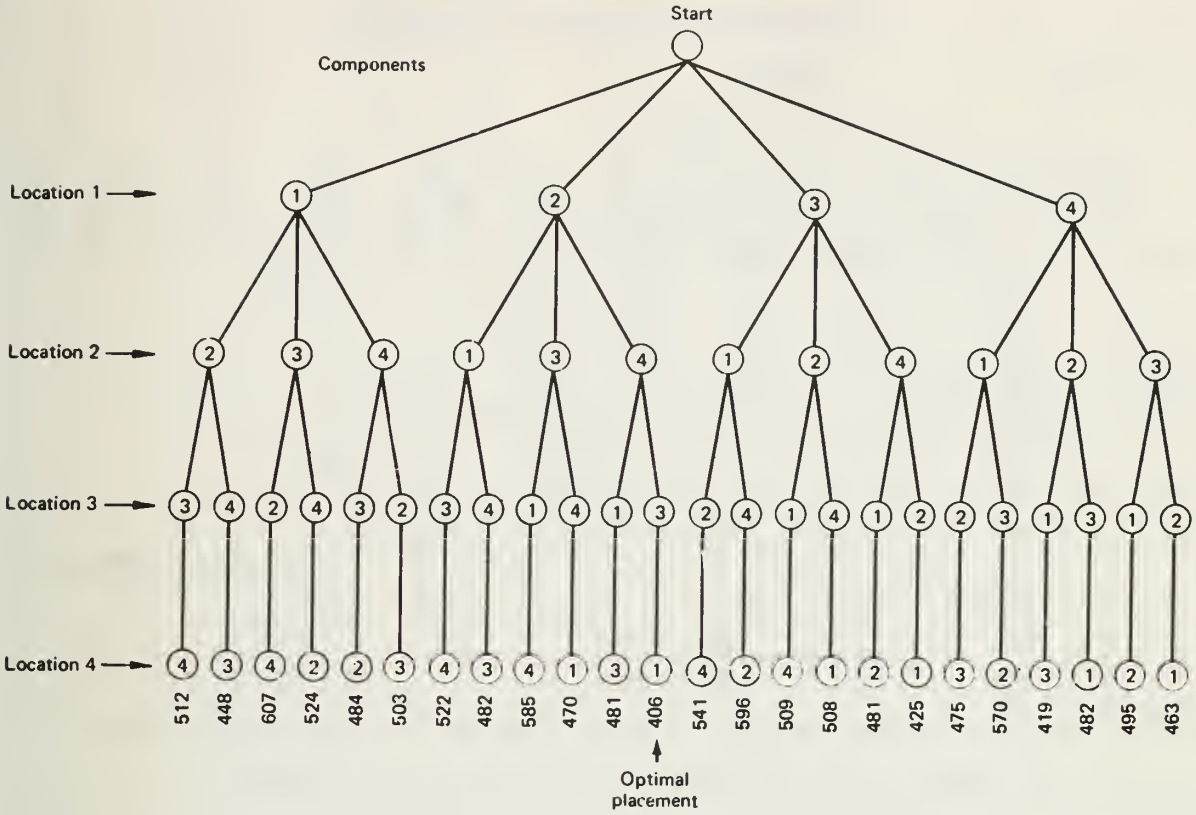


FIGURE 8. Complete enumeration of $n!$ feasible solutions

TABLE 1. *2-VBB With Minimum Permuted Dot Product Bounds*

Level 1				Level 2				Level 3				Level 4				
Component i	Location j	B^1_{ij}	Minimum	Component i	Location j	B^2_{ij}	Minimum	Component i	Location j	B^3_{ij}	Minimum	Component i	Location j	B^4_{ij}	Optimal
1	1	437													
1	2	467		1	2	467									
1	3	419		1	3	481									
1	4	395		1	4	395	*								
2	1	395	*												
2	2	419													
2	3	403													
2	4	451													
3	1	416													
3	2	450		3	2	465		3	2	466					
3	3	406		3	3	406		3	3	406		3	3	406	*
3	4	410		3	4	457									
4	1	410													
4	2	406		4	2	406		4	2	401	*				
4	3	430		4	3	430		4	3	470					
4	4	486		4	4	522									

TABLE 2. *2-VBB With Associated Mean Bounds*

Level 1				Level 2				Level 3				Level 4			
Component i	Location j	B_{ij}^1	Minimum	Component i	Location j	B_{ij}^2	Minimum	Component i	Location j	B_{ij}^3	Minimum	Component i	Location j	B_{ij}^4	Optimal
1	1	513													
1	2	531													
1	3	495													
1	4	459	*												
2	1	491		2	1	438		2	1	406	*				
2	2	479		2	2	495									
2	3	502		2	3	444		2	3	425					
2	4	525													
3	1	510		3	1	466		3	1	425					
3	2	524		3	2	466									
3	3	496		3	3	444		3	3	406		3	3	406	*
3	4	468													
4	1	484		4	1	472									
4	2	463		4	2	415	*								
4	3	504		4	3	489									
4	4	546													

TABLE 3. *2-VBB With Maximum Permuted Dot Product Bounds*

1	1	607		1	1	503									
1	2	596													
1	3	585		1	3	481									
1	4	508		1	4	410	*								
2	1	590		2	1	481		2	1	406	*				
2	2	533													
2	3	607		2	3	503		2	3	425					
2	4	601		2	4	494									
3	1	601		3	1	490		3	1	425					
3	2	607													
3	3	580		3	3	484		3	3	406		3	3	406	*
3	4	523		3	4	503									
4	1	570													
4	2	503	*												
4	3	576													
4	4	607													

Another set of experiments concerned the trial solution of the problem with the 1-VBB algorithm using each of the three bounding processes. Table 4 records the statistics for the application of the minimum permuted dot product or Gilmore algorithm. The optimal solution was achieved using the location to order pairing of (1, 2, 3, 4) as opposed to the Gilmore "rule of thumb" ordering (2, 1, 3, 4). This same location to level ordering produced inferior solutions as demonstrated in Tables 5 and 6 with the other bounding alternatives. The optimal solutions were obtained by using a different preassignment

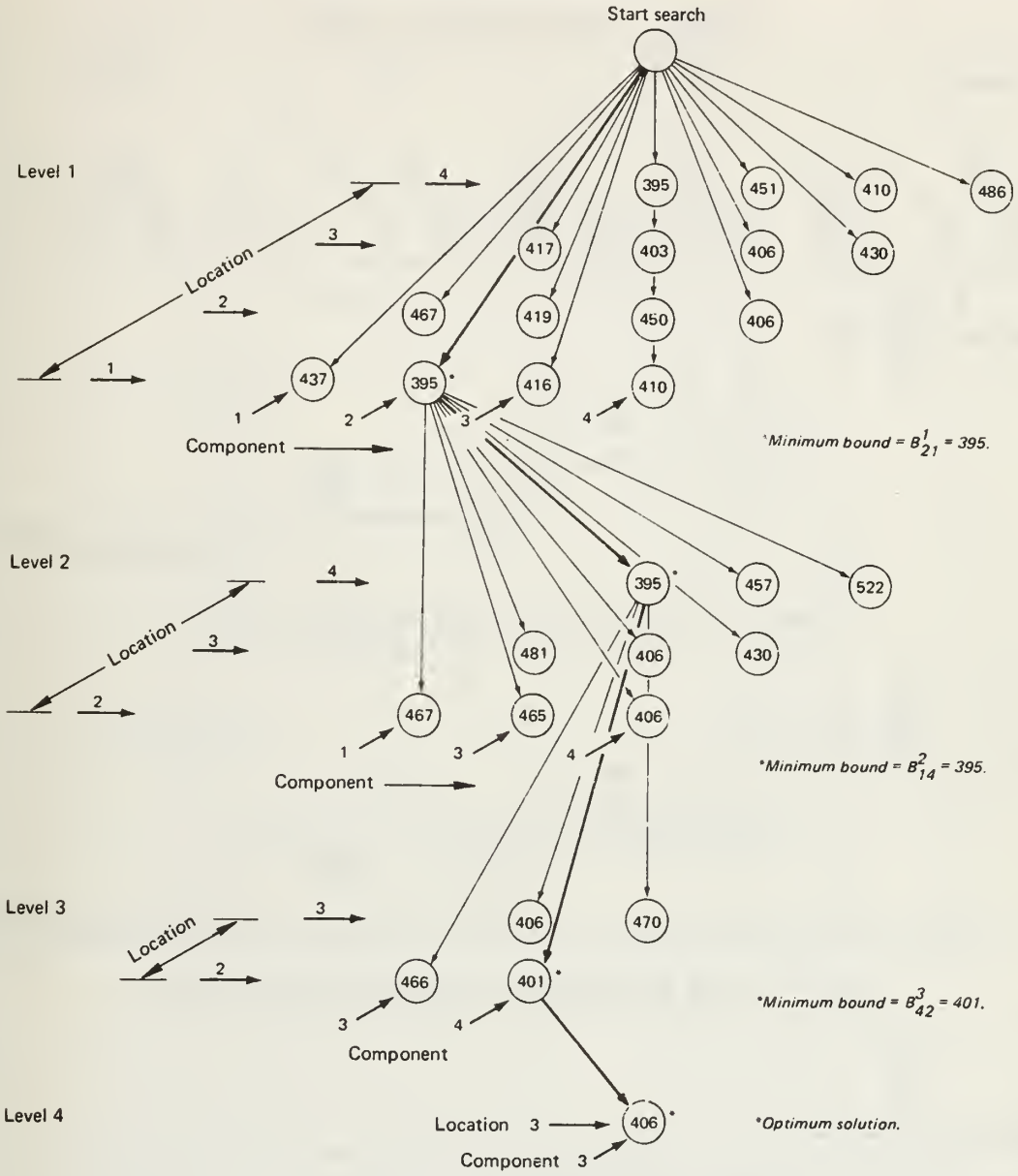


FIGURE 9. Three-dimensional search tree (Gavett-Plyter)

in each case. The determination of the proper order must, unfortunately, be found by "trial and error" methods. The two-dimension search-tree can be visualized by inspecting the statistics of Tables 4, 5, and 6. The statistics of Table 5 are shown as a two-dimensional search-tree in Figure 10. At level 1, four component choices are possible for pairing with location 1 with the selection of component 2 being made on the basis of the minimum bound 395. At level 2, three component selections are possible for permanent assignment to location 2. Component 4 with the minimum level 2 bound 406 is chosen. At level 3, component 3 is matched with location 3 (minimum bound 406) and at level 4, component 1 is assigned to location 4. The final bound 406 is also the final value of the objective function $Z(S')$. The final solution vector S' is then (2, 4, 3, 1).

TABLE 4. *Gilmore Algorithm (1-VBB)*

Level 1				Level 2				Level 3				Level 4			
Component i	Location j	B^1_{ij}	Minimum	Component i	Location j	B^2_{ij}	Minimum	Component i	Location j	B^3_{ij}	Minimum	Component i	Location j	B^4_{ij}	Optimal
1	1	437													
2	1	395	*												
3	1	416													
4	1	410													
				1	2	467									
				3	2	465									
				4	2	406	*								
								1	3	476					
								3	3	406	*				
												1	4	406	*

TABLE 5. *1-VBB With Value of Associated Mean Bound*

1	1	513													
2	1	491													
3	1	510													
4	1	484	*												
				1	2	509									
				2	2	448	*								
				3	2	467									
								1	3	453	*				
								3	3	466					
												3	4	419	*

TABLE 6. *1-VBB With Maximum Permuted Dot Product Bounds*

1	1	607													
2	1	590													
3	1	601													
4	1	570	*												
				1	2	570									
				2	2	482	*								
				3	2	508									
								1	3	419	*				
								3	3	482					
												3	4	419	*

The Steinberg Test Problem

An experiment was conducted using the larger and more complex problem originally reported by Steinberg [15]. This problem is reported to be an actual computer backboard layout for a UNIVAC computer. This problem is concerned with attempting to find the optimal assignment of 34 component modules to 36 backboard locations which minimizes the total interconnecting wire length among components. This problem is presented in order to provide a quantitative basis for comparing the various algorithms.

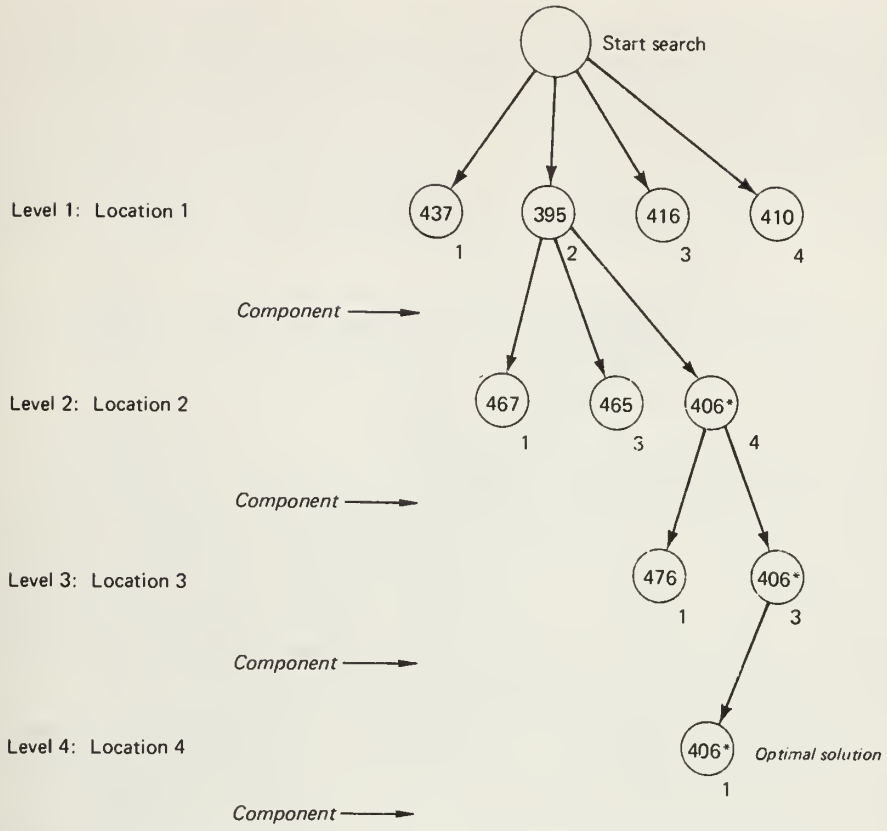


FIGURE 10. Two-dimensional search tree (Gavett-Plyter)

The F matrix for the components is given in Figure 11. A symmetrical matrix was formed by

$$\sum_i (f_{ij}+f_{ji}) \qquad (i,j=1,\ldots,n)$$

and only the part above the diagonal was retained. This particular F matrix is sparse, that is, about 70-percent zeros. The number of interconnecting wires varies from 0 to 316 with mean equal to 15.26. The extreme variation in F creates a large range in potential solution values. For example, the minimum possible (infeasible) solution value is 3001 and the mean of all possible solution values is 9378.58. Therefore, the minimum feasible solution is located between these values, that is, 3001 < minimum $Z \ll 9378$, most likely much closer to 3001 than 9378. On the average each component is connected to 10 other components with the maximum being 26 and the minimum 1. The graph which results from considering the components as nodes and the interconnections as links is continuous in that no disjoint subsets exist. Thus, the problem cannot be decomposed into several independent problems of lesser size. A total of 2625 wires connect the components.

The circuit board has 36 positions arranged to form a 4×9 grid as illustrated in Figure 12. The distance matrix is constructed by determining the two-dimensional Euclidean distance; that is,

$$d_{jq} = \sqrt{(x_j - x_q)^2 + (y_j - y_q)^2}.$$

	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10	E11	E12	E13	E14	E15	E16	E17	E18	E19	E20	E21	E22	E23	E24	E25	E26	E27	E28	E29	E30	E31	E32	E33	E34	Total	↓
E1				2	1	7	9		4	75	7	12	22	7	1					23															170	12
E2							4	16		8			16					6		4															54	6
E3						4	16	20					20							4															64	5
E4					29	5	18	47	23	2	4		48		4					25															207	11
E5						18	12	25			4		25		3					18		3													138	10
E6							4	2		1	23		2	19						2	19														106	12
E7									14	72	7	8	39	8	40	8		8	4	7								28	8						314	19
E8									10	71	2							41										7	8						249	11
E9										14			18																						83	6
E10											11	1	17		1			17		15															305	13
E11											316	33	8	2					8	34			6				10			6					481	16
E12												157	25	4				1						22		1									549	11
E13													11	6				6		5	8	3	10				9	11	2			1			486	22
E14														3				1	1	21		1		2			5			3	2	5	5	4	112	17
E15															19			2	2	12								7	3						109	15
E16																		6		1															34	4
E17																		40																	40	1
E18																				26															154	11
E19																				13	9		7					27	16	3		20		4	116	13
E20																					11	4	36				16	18	9	10	1	28	6	2	368	26
E21																						36	6		8		2								80	7
E22																											4								51	6
E23																										12	9								86	7
E24																									26		5								33	3
E25																										35	2								93	5
E26																											4								51	3
E27																																			74	12
E28					Total number of wires: 2625																								10	22	4	6	4	12	157	12
E29																													19	12					79	8
E30																														19	4	5	8		99	10
E31																															3	13			54	7
E32																															18	24			106	8
E33																																20			61	7
E34																																			87	8

FIGURE 11. Component interconnection matrix for Steinberg problem

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36

FIGURE 12. 4×9 circuit board configuration

The distance from location 1 to location 36 in Figure 12 is for example

$$\sqrt{3^2 + 8^2} \sim 8.54,$$

units.

Four branch and bound trials were conducted on the Steinberg problem. First, the 1-VBB algorithm was evaluated using each of the three bounding processes. This was done in an effort to duplicate and validate the results reported in the literature. Then the 2-VBB algorithm using only the value of the associated mean bound was considered. The reason for excluding the other two bounding processes was based on the unacceptable large execution estimates.*

The performance statistics for each of the four trials are shown in Table 7. The published results

TABLE 7. *Branch and Bound Algorithms Performance Statistics*

Algorithm	Performance statistics			
	This study		Literature	
	Solution	Time (seconds)	Solution	Time (seconds)
Single-variable branch and bound				
A. Gilmore	4863.41	*88.5	4547.54	~60
B. Mean value rule	5039.38	*4.8	—	—
C. Maximum permuted dot product	4696.00	*90.0	—	—
2-variable branch and bound				
D. Graves and Whinston ($N \times M$ search tree)	4612.27	**35.0	4612.27	***<1.0
E. Improved algorithm ($N \times N$ search tree)	4419.49	**37.0	—	—

*IBM 360/75
**CDC 3800
***IBM 360/91

of Gilmore could not be duplicated because the order of the location to level assignments was not explicitly stated. The results of Graves and Whinston were duplicated by solving an $N \times M$ problem.

The improved 2-VBB algorithm produced the best solution (4419.49) in a time much less than that required by the Gilmore 1-VBB algorithm. The final placement for the 2-VBB algorithm is shown in Figure 13.

		15	16	14	29	30	33	31
3	6	1	13	12	28	32	34	22
2	18	10	7	11	20	19	21	24
17	9	8	4	5	27	23	25	26

FIGURE 13. Placement for 2-VBB (4419.49)

* Approximately 30 min as compared to ~ 10 sec for the value of the associated mean bound.

Considerable emphasis was placed on program optimization to achieve the execution times listed in Table 7. For example, through the use of efficient indexing schemes, the Gilmore algorithm's execution time was reduced from 240 to 88 secs. This was accomplished by transforming the objective function into the following:

$$\text{minimize } Z = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} f_{ij} d_{1(i)1(j)},$$

where f_{ij} is a symmetric matrix. This transformation reduces the number of multiplications and additions required to evaluate the objective function from n^2 to $((n^2-n)/2)$. This indexing scheme was also used in the bounding processes to reduce computational effort. Another optimization procedure made use of the low density factor of the F matrix (i.e., ~30 percent). Here all zero term multiplications were not performed in forming either the min or max permuted dot product bounds.

VI. EVALUATION OF "NEAR-OPTIMAL" PLACEMENT ALGORITHMS

In order to address the question of just how good are "near-optimal" solutions, a sampling experiment was conducted to characterize and elucidate the solution space. Permutations of length n (36) were drawn at random from the population of all possible feasible assignment vectors. In this experiment, 10,000 samples were made and evaluated. Although seemingly a large sample size, 10,000 represents a mere fraction of the 3.7199×10^{41} (36!) possible feasible solutions to the Steinberg test problem.

The cumulative distribution curve of the 10,000 samples is shown in Figure 14. The mean of the

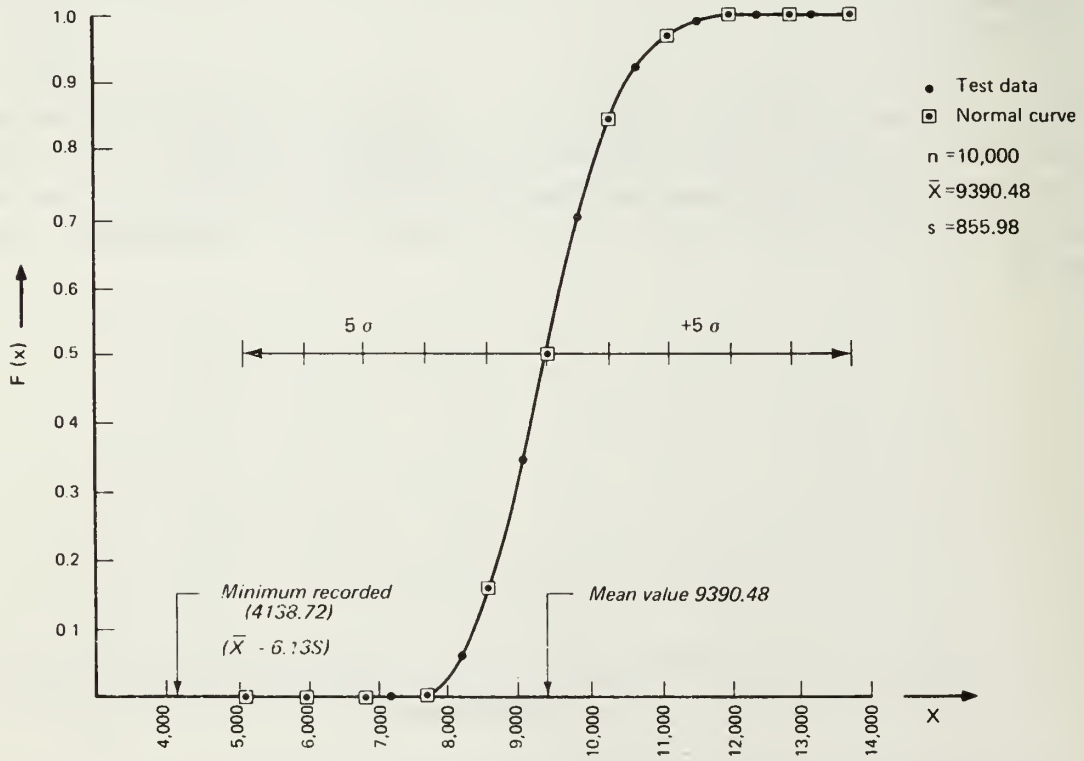


FIGURE 14. Cumulative distribution of 10,000 random samples

sample is $\bar{x}=9390.48$ and the sample standard deviation $s=855.98$. The sample minimum was 6923.06 and maximum 12,528. The sample curve is shown superimposed over a normal cumulative distribution function with parameters (μ, σ) equal to the sample statistics. It can be observed that the sample distribution appears to be nearly normal. Subsequent experiments were conducted with smaller sample sizes. These experiments verified that a solution near $(\mu - 3\sigma)$ could be achieved through random (stochastic) enumeration with very few iterations ($\sim$ 1,000); however, the probability of any major improvement in subsequent iterations is very remote.

The magnitude of this likelihood can be estimated using the assumption of a normally distributed population.

With this assumption then, the probability that a solution, selected at random from the normally distributed population, falls $k\sigma$ *below* the population mean is given by

$$\text{prob}\{x \leq (\mu - k_\sigma)\} = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\mu - k\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

For example, when $k=5.0$ this probability is equal to

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\mu - 5.0} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0.0000002$$

or approximately 1 chance in 5,000,000.

Some insight into the difficult nature of the placement problem can now be gained by considering previously reported solutions to this problem in light of the above sampling experiment. The best solution obtained to date is 4138.72* which is 6.13s below the sample mean. The probability of finding this solution by random search is less than 1 chance in 1 billion. In fact, all of the branch and bound algorithms considered were capable of producing solutions which were less than 5σ below the mean.

VII. SUMMARY AND CONCLUSIONS

1. An improved placement procedure, the two-variable, n-step, tree search algorithm, was introduced and was demonstrated to be effective in solving the component placement problem. The salient feature of this algorithm is the ability to consider the placement of both component and location simultaneously. The result is that the two-variable algorithm will produce closer to optimal solutions which are completely reproducible at a reasonable cost.

2. Search-tree techniques in general were demonstrated to be both a powerful and an efficient means for determining near optimal solutions to the quadratic assignment problem. In the case of the Steinberg problem, the solutions found were at least among the 0.00002-percent level of the total population of all feasible solutions based on the assumption of normality. This estimate is extremely conservative since an infinite range on the distribution space is assumed rather than a truncated distribution function of the actual finite solution space.

---

* Found by applying a pair-exchange algorithm to the Gaschutz and Ahrens solution in Reference [3].

REFERENCES

[1] Armour, G. C. and E. S. Buffa, "A Heuristic Algorithm and Simulative Approach to Relative Location of Facilities," Management Science Vol. 9, No. 2 (Jan. 1963), p. 294–309.

[2] Breuer, M. A., "The Formulation of Some Allocation and Connection Problems as Integer Programs," Nav. Res. Log. Quart. *13*, 83–95 (Mar. 1966).

[3] Gaschutz, G. K. and J. H. Ahrens, "Suboptimal Algorithms for the Quadratic Assignment Problem," Nav. Res. Log. Quart. *15*, 49–62 (Mar. 1968).

[4] Gavett, J. W. and N. V. Plyter, "The Optimal Assignment of Facilities to Locations by Branch and Bound," Operations Research, Vol. 14, No. 2 (Mar.–Apr. 1966), 210–232.

[5] Gilmore, P. C., "Optimal and Suboptimal Algorithms for the Quadratic Assignment Problem," Journal Soc. for Industrial and Applied Mathematics Vol. 10, No. 2 (June 1962), p. 305–313.

[6] Graves, G. W. and A. B. Whinston, "An Algorithm for the Quadratic Assignment Problem," Management Science Vol. 17, No. 7 (Mar. 1970).

[7] Hillier, F. S., "Quantitative Tools for Plant Layout Analysis," Journal of Industrial Engineering Vol. 14, No. 1 (Jan.–Feb. 1963). p. 33–40.

[8] Hillier, F. S. and M. M. Connors, "Quadratic Assignment Problem Algorithms and the Location of Indivisible Facilities," Management Science Vol. 13, No. 1 (Sept. 1966), p. 42–57.

[9] Koopmans, T. C. and M. Beckmann, "Assignment Problems and the Location of Economic Activities," Econometrica Vol. 25, No. 1 (Jan. 1957), p. 53–76.

[10] Lawler, E. L., "The Quadratic Assignment Problem," Management Science Vol. 9, No. 4 (July 1963), p. 586–599.

[11] Little, J. D. C., K. G. Murty, D. W. Sweeney, and K. Caroline, "An Algorithm for the Traveling Salesman Problem," Operations Research Vol. 14, No. 6 (Nov.–Dec. 1963), p. 972–989.

[12] Nakahara, H., "Computer-Aided Interconnection Routing: General Survey of the State-of-the-Art," Networks Vol. 2, No. 2 (1972), p. 167–183.

[13] Nugent, C. E., T. E. Vollman, and J. Ruml, "An Experimental Comparison of Techniques for the Assignment of Facilities to Locations," Operations Research Vol. 16, No. 1 (Jan.–Feb. 1968), p. 150–173.

[14] Pierce, J. F. and W. B. Crowston, "Tree-Search Algorithms for Quadratic Assignment Problems," Nav. Res. Log. Quart. *18*, 1–36 (Mar. 1971).

[15] Steinberg, L., "The Backboard Wiring Problem: A Placement Algorithm," Soc. for Ind. and Applied Mathematics, Vol. 3, No. 1 (Jan. 1961), p. 37–50.

PARAMETRIC LINEAR PROGRAMMING: SOME SPECIAL CASES

W. Dent
University of Iowa

R. Jagannathan
Columbia University

M. R. Rao
University of Rochester

1. INTRODUCTION

The parametric linear programming problem in which the coefficient matrix is parameterized has been studied by several authors including Saaty [5], Courtillot [2], Willner [8] and Barnett [1]. We consider the general problem of the coefficient matrix being parameterized by a matrix of rank k and show that the parametric program so defined is equivalent to a problem in which only k coefficients depend on the parameter. Thus, for the special case of $k=1$ (Willner [8]) the problem simplifies to solving a linear program in which only one coefficient depends on this parameter (Simonnard [6]).

2. PROBLEM FORMULATION

The parametric linear program under consideration can be stated as

$$\text{Max } c'x \tag{1}$$

subject to

$$(A+\alpha R)x=b \tag{2}$$

$$x \geq 0; \qquad 0 \leq \alpha \leq \bar{\alpha}$$

where A and R are $m \times n$ matrices;

c, x and b are respectively $n \times 1$, $n \times 1$ and $m \times 1$ column vectors;

α is a scalar parameter.

We also assume that rank $(A)=m$ and rank $(R)=k \leq m$. The main purpose of this section is to reduce the parametric program (1) to one in which only k elements involve the parameter α. We discuss below two procedures for accomplishing this.

METHOD 1:

We will adopt the following notation:

R_i: The ith row of R, $i=1, \ldots, m$

$\bar{R}$: The $k \times n$ submatrix of R comprising the first k rows of R.

R^* : The $(m-k) \times n$ submatrix of R comprising the last $(m-k)$ rows of R, so that $R = \begin{bmatrix} \bar{R} \\ R^* \end{bmatrix}$

$\bar{A}$: The $k \times n$ submatrix of A consisting of the first k rows of A.

A^* : The $(m-k) \times n$ submatrix of A consisting of the last $(m-k)$ rows of A so that $A = \begin{bmatrix} \bar{A} \\ A^* \end{bmatrix}$

$\bar{b}$: The $k \times 1$ subvector of b consisting of the first k elements of b.

b^* : The $(m-k) \times 1$ subvector of b consisting of the last $(m-k)$ elements of b, so that $b = \begin{bmatrix} \bar{b} \\ b^* \end{bmatrix}$

Suppose without loss of generality that the first k rows $R_1, R_2, \ldots, R_k$ are linearly independent. Then there exists an $(m-k) \times m$ matrix L such that $R^* = L\bar{R}$.

Consider the following parametric linear program:

(3) $$\text{Max } c'x$$

subject to

$$\bar{A}x + \alpha y = \bar{b}$$

$$\bar{R}x - y = 0$$

$$(A^* - L\bar{A})x = (b^* - L\bar{b})$$

$$x \geq 0; \qquad 0 \leq \alpha \leq \bar{\alpha}$$

THEOREM 1: Problem (3) is equivalent to Problem (1).

PROOF: Note that Constraint (2) can be replaced by an equivalent set of constraints

$$\bar{R}x - y = 0$$

(4) $$\bar{A}x + \alpha y = \bar{b}$$

(5) $$A^*x + \alpha L y = b^*$$

Eliminating αy in (5) by using (4), we have the required result.

This completes the proof.

METHOD 2:

Let us adopt the following notation:

Q_j : The jth column of R, $j = 1, \ldots, n$.

$\bar{Q}$: The $m \times k$ submatrix of R comprising the first k columns of R.

Q^* : The $m \times (n-k)$ submatrix of R comprising the last $(n-k)$ columns of R, so that $R = [\bar{Q}, Q^*]$.

Suppose without loss of generality that the column vectors $Q_1, Q_2, \ldots Q_k$ are linearly independent. Then, there exists a $k \times (n-k)$ matrix M such that $Q^* = \bar{Q}M$. Therefore, $R = [\bar{Q}, \bar{Q}M] = \bar{Q}[I, M]$. Let $V = [I, M]$ and let V_i, $i = 1, \ldots, k$, be the ith row vector of V. Then $R = \sum_{i=1}^{k} Q_i V_i$, which is incidentally a decomposition of R into k matrices, each of which is of rank one.

Consider a parametric linear program:

(6) $$\text{Max } c'x$$

subject to

$$V_i x - \beta y_i = 0, \qquad 1 = 1, \ldots, k$$

$$Ax + \sum_i Q_i y_i = b$$

$$x \geq 0; \qquad \beta \geq 1/\bar{\alpha}$$

where $\beta = 1/\alpha$.

THEOREM 2: Problem (6) is equivalent to Problem (1).

PROOF: Constraint (2) can be replaced by

(7) $$[A + \alpha \Sigma Q_i V_i]x = b.$$

Setting

(8) $$V_i x - \beta y_i = 0$$

and substituting in (7) yields

$$Ax + \alpha\beta \Sigma Q_i y_i = b.$$

Since $\alpha\beta = 1$, we have the required result. This completes the proof.

REMARKS:

(1) Problems (3) and (6) both have $(n+k)$ variables and $(m+k)$ constraints with the y variables unconstrained. Further, both problems have k parameterized coefficients (Dinkelbach [3]). In solving the parametric linear program none of the variables y_i may be considered as a candidate for exit from the basis. Alternatively, we may replace y_i by two non-negative variables y_i^+ and y_i^-, so that $y_i = y_i^+ - y_i^-$.

(2) For the special case of $k = 1$, both problems (3) and (6) have $(n+1)$ variables, $(m+1)$ constraints, and only one coefficient is parameterized. The computational details discussed in [6, Section 7.10] are applicable here with the modification that the variable γ is not allowed to leave the basis. Thus, the proposed solution procedure is considerably simpler than the algorithm of Willner [8] for this problem. See also [4, 7].

REFERENCES

[1] Barnett, S. "A Simple Class of Parametric Linear Programming Problems," Operations Research, Vol. 16, No. 6 (Nov.–Dec., 1968), pp. 1160–1165.

[2] Courtillot, M. "On Varying all the Parameters in a Linear Programming Problem and Sequential Solution of a Linear Programming Problem," Operations Research, Vol. 10, No. 4 (July–Aug., 1962), pp. 471–475.

[3] Dinkelbach W. *Sensitivitätsanalysen Und Parametrische Programming*, Springer-Verlag, Berlin-New York, 1969.

[4] Kim, C. "Parameterizing an Activity Vector in Linear Programming," Operations Research, Vol. 19, 1632–1646 (1971).

[5] Saaty, T. L. "Coefficient Pertubation of a Constrained Extremum," Operations Research, Vol. 17, No. 3 (May–June, 1959), pp. 294–302.

[6] Simonnard M. *Linear Programming*, Prentice Hall, Inc., Englewood Cliffs, N.J.

[7] Van de Panne C. "Parameterizing an Activity Vector in Linear Programming," Operations Research, *21*, 389–390 (1973).

[8] Willner, L. B. "On Parametric Linear Programming," SIAM J. Appl. Math., Vol. 15, No. 5 (Sept., 1967), pp. 1253–1257.

SEQUENTIAL SEARCH OF AN OPTIMAL DOSAGE: NON-BAYESIAN METHODS*

B. H. Eichhorn

Case Western Reserve University

1. INTRODUCTION

The present research represents a class of one-sided (sequential) stochastic approximation procedures, which have possible applications in many fields. A motivational background could be simply described in terms of a biomedical problem. A drug could be administered at various dosages. High dosages although desirable increase the level of toxicity. The dosage-toxicity relationship is not completely known. We want to use sequential experimentation to approach from below the highest possible dosage one may give without causing too much toxicity. For convenience let us use this biomedical terminology.

Let x designate an assigned dosage. This variable can be controlled by the experimenter. Let $Y(x)$ designate an observable random variable, which is the toxicity level associated with the dosage x. The distribution function of $Y(x)$ depends on x in a manner specified later under the various statistical models. Generally, the expected toxicity is an increasing function of the dosage. A threshold of toxicity is specified in the sense that higher toxicity levels are undesirable. For the sake of simplicity, assume that the toxicity threshold assumes the value zero. The objective is to assign the largest possible dosages without crossing the threshold of toxicity. Since toxicity levels are not determined completely by the dosages, a tolerance probability γ, $0 < \gamma < 1$, is specified, and the optimal dosage, ξ_γ, is defined as the largest x value under which the probability that $Y(x) \leq 0$ is not smaller than γ; i.e.,

$$P[Y(\xi_\gamma) \leq 0] \geq \gamma. \tag{1.1}$$

If ξ_γ were known it would have been used in each case. In certain models one could make simple transformations and consider instead of ξ_γ the value of x for which $E\{Y(x)\} = 0$. This value is denoted by ξ. In all the models treated here both ξ and ξ_γ exist and are unique. In the remainder of the present section we use ξ to designate the optimal dosage in either one of the above cases.

A sequential search procedure is a procedure of determining a sequence of dosages $x_1, x_2, \ldots$; where for every $n > 1$ x_n is a measurable function of $(x_1, \ldots, x_{n-1})$ and of $(Y(x_1), \ldots, Y(x_{n-1}))$. We consider procedures which satisfy the following two conditions:

(i) *Feasibility*—For each n, $n = 1, 2, \ldots$, and a preassigned sequence $\{\alpha_n;\ n \geq 1\}$, where $0 < \alpha_n < 1$,

$$P[x_n > \xi] \leq \alpha_n. \tag{1.2}$$

* Partially supported by Project NR 042-276 of the Office of Naval Research at Case Western Reserve University.

(ii) *Consistency*—x_n converges in probability to ξ. If almost sure convergence holds we speak of strong consistency.

The value of α_n represents a bound on the probability of exceeding the optimal dosage ξ at the nth stage. The values of α_n can be held fixed, or can be decreased as information on ξ increases. In special cases α_n can be chosen so that $\sum_{n=1}^{\infty} \alpha_n \leq e$, without losing consistency. In such cases the overall probability of exceeding ξ anywhere in the sequence is smaller than e. We further notice that at the nth stage, the probability that the observed toxicity $Y(x_n)$ will exceed the threshold is bounded by $(1-\alpha_n)(1-\gamma)+\alpha_n$. In the present paper we restrict attention to the case of fixed α.

There are many feasible and consistent sequential search procedures for any one of the considered problems, and it is desirable to introduce some optimality criteria. Due to the complexity of the problem we do not set rigid optimality conditions. However, some of the procedures that will be presented have certain optimal properties, which will be discussed in the sequel. In a recent paper [1] the search procedure was studied for the special cases of linear regression $E\{Y(x)\}=a+bx$, with a conditional normal distribution of $Y(x)$ at each x. In that paper the intercept and variance were assumed to be known. Non-Bayesian and Bayesian procedures were develped. The present paper provides solutions for cases in which either the slope or the intercept is known and the variance may be know or unknown. Only non-Bayes procedures are discussed here. In section 2 the statistical model is explicitly specified. The case of unknown intercept is discussed in section 3. Section 4 is devoted to the case of unknown variance. The case of both intercept and variance unknown is discussed in section 5. Section 6 treats the case of both slope and variance unknown. The general case of linear regression in which all the parameters are unknown is subject for further research.

2. THE STATISTICAL MODEL

The following assumptions are kept throughout this paper. Let $Y(x)$ designate a random variable representing the observed response at dosage x. We assume that the conditional distribution of $Y(x)$, given x, is normal with mean $h(x)$ and variance $\sigma^2(x)$. We further assume that $h(x)$ is a linear function, $h(x)=a+b\,x$, with $a<0$ and $b>0$ over the interval of interest $0\leq x\leq k$. The optimal dosage ξ (or ξ_γ) belongs to this interval. With respect to the variance $\sigma^2(x)$ we distinguish between two models:

Model I; $\sigma^2(x)=x^2\sigma^2$, and

Model II; $\sigma^2(x)=\sigma^2$; where σ^2 is a positive constant.

For the sake of simplicity we assume that the maximum allowable toxicity is $\eta=0$.

A general formula for the optimal dosage ξ_γ is given by:

$$\xi_\gamma=\begin{cases}-a/b+z_\gamma\sigma), & \text{in Model I}\\ -(a+z_\gamma\sigma)/b, & \text{in Model II;}\end{cases} \tag{2.1}$$

where z_γ is the γ-fractile of the standard normal distribution. In cases of Model II, with *known* σ we can further simplify by making the transformation $\eta\rightarrow\eta-z_\gamma\sigma$ again taking it to be zero. In this case ξ_γ assumes the simple expression $-a/b$, which will be denoted by ξ.

3. CASES OF UNKNOWN INTERCEPTS

We assume here that the slope b and variance σ^2 are known.

3.1. Model I

We start the study of cases with unknown intercepts with Model I. Here we have to assume the knowledge of a value x^* such that $0 < x^* \leq \xi_\gamma$. A search procedure is defined as a sequence X_1, X_2, . . . of dosages each being a function of all former dosages and observations of toxicity. For Model I we define the search procedure $P.1.$ as follows:

Define:

$$Y_i = Y(X_i), \qquad i = 1, 2, \ldots$$

$$U_i = Y_i - bX_i, \qquad i = 1, 2, \ldots$$

$$\bar{U}_n = \sum_{i=1}^{n} U_i/n, \qquad n = 1, 2, \ldots \tag{3.1}$$

$$\hat{\xi}_n = -(\bar{U}_n + \sigma k/\sqrt{n\alpha})/(b + z_\gamma \sigma), \qquad n = 2, 3, \ldots$$

and

$$\hat{\xi}_1 = -(U_1 + \sigma x^*/\sqrt{\alpha})/(b + z_\gamma \sigma).$$

The initial dosage is taken to be

$$X_1 = x^*,$$

and successively we set

$$X_{n+1} = \text{Max}\,(x^*, \hat{\xi}_n), \qquad n = 2, 3, \ldots. \tag{3.2}$$

THEOREM 1: *The search procedure* P.1. *is feasible and consistent*

PROOF: The random variables U_i, $i = 1, 2, \ldots$ have conditional normal distributions, given x_i, with mean a and variances $x_i^2\sigma^2$.

These variables are uncorrelated, but not independent. Indeed for every $i < j$:

$$E\{U_iU_j\} = E\{U_iE[U_j|\mathscr{F}_i]\} = E\{U_ia\} = a^2, \tag{3.3}$$

where $\mathscr{F}_i$ denotes the σ-field generated by the first i random variables. Thus, $E\{\bar{U}_n\} = a$ and the variance of $\bar{U}_n$ is bounded by σ^2k^2/n. Hence, from the Chebychev inequality we infer that $\hat{\xi}_n$ *is a* $1-\alpha$ *lower confidence limit for* $\xi_\gamma = \dfrac{-\,a}{b + z_\gamma\sigma}$. Furthermore, by assumption $x^* \leq \xi_\gamma$. We therefore conclude that the procedure $P.1.$ is feasible, i.e.,

$$P(X_{n+1} \leq \xi_\gamma) \geq 1 - \alpha \qquad \text{for each } n \geq 0.$$

The consistency of $P.1.$ is established from the fact that $\text{Var}\,(\bar{U}_n) \leq \sigma^2k^2/n$. Hence $\bar{U}_n$ converges in probability to a, and $\sigma k/\sqrt{n\alpha} \to 0$ as $n \to \infty$, · Hence $\hat{\xi}_n \to \xi$ in probability. (Q.E.D.)

We remark that the procedure $P.1.$ is based on a rather crude inequality, and the upper limit k might be very large. Thus, although the procedure has the required properties it may be inefficient for small values of n. The efficiency of the procedure could be improved by further investigating and employing the properties of the distribution of $\bar{U}_n$.

3.2. Model II

Whenever Model II can be assumed, i.e., $\sigma^2(x)=\sigma^2$ for all x, we can attain stronger results. Let $U_i \cdot i=1, 2, \ldots$ and $\bar{U}_n n=1, 2, \ldots$ be defined as in Model I, and set

$$\hat{\xi}_n=-\bar{U}_n/b-\sigma z_{1-\alpha}/b\sqrt{n}, \qquad n=1, 2, \ldots. \tag{3.4}$$

The sequence of dosages under search procedure $P^*.1.$ is determined in the following manner:

The first dosage is $x_1=0$, and for every $n=2, 3, \ldots$ the dosages, x_n, are specified by

$$x_n=\text{Max}(, \hat{\xi}_{n-1}) \tag{3.5}$$

THEOREM 2: *The search procedure* P*.1. *is feasible and strongly consistent.*

PROOF: U_i, $(i=1, 2, \ldots, n)$ are independent identically distributed (i.i.d.) random variables. They have a normal distribution with mean a and variance σ^2. Since a is negative ξ is positive, and $x_1<\xi$. Furthermore, $\bar{U}_n$ is normally distributed like $N(a, \sigma^2/n)$. Hence $\hat{\xi}_n$ is a $1-\alpha$ lower confidence limit for $\xi=\frac{-a}{b}$. Moreover, $\hat{\xi}_n$ is a uniformly most accurate lower confidence limit (U.M.A.; see Lehmann [2], pp. 78–81). This property will be further discussed later. The strong consistency of $P^*.1.$ is shown as follows: We have $\bar{U}_n \xrightarrow{a.s.} a$, and $\sigma z_{1-\alpha}/\sqrt{n} \to 0$ as $n \to \infty$. Therefore $\hat{\xi}_n=-\bar{U}_n/b-\sigma z_{1-\alpha}/b\sqrt{n} \xrightarrow{a.s.} -a/b=\xi$, and $x_{n+1}=\text{Max}(0, \hat{\xi}_n) \xrightarrow{a.s.} \xi>0$. (Q.E.D.)

The procedure $P^*.1.$ has a certain optimality property that will be discussed now. Define the dosage "shortage" at the nth trial as $R_n=(\xi-X_n)^+$, which is the positive part of the distance from X_n to the optimal dosage ξ.

A procedure will be called *optimal* if it is feasible and for each N, $N=1, 2, \ldots$ it minimizes the expectation of the total shortage in the first N steps, i.e.,

$$E\left(\sum_{i=1}^{N} R_i\right).$$

THEOREM 3: *The procedure* P*.1. *is optimal.*

PROOF: We shall show that for each n, $n=1, 2, \ldots$, the procedure $P^*.1.$ minimizes the expected shortage R_n, with respect to all feasible procedures. Since this property holds at each stage independently of what happens at all other stages the optimality of $P^*.1.$ will follow.

Consider the nth trial, and suppose that an arbitrary feasible procedure $\hat{\Pi}$ assigns it a dosage $\hat{x}_n$. $\hat{x}_n$ is a function of the first $(n-1)$ observations only. Since $\hat{\Pi}$ is feasible, $\hat{x}_n$ is a lower confidence limit for ξ at level $(1-\alpha)$. Hence, the UMA property of $\hat{\xi}_{n-1}$ implies that $P_\xi\{\hat{\xi}_{n-1} \leq \xi'\} \leq P_\xi\{\hat{x}_n \leq \xi'\}$ for all $0<\xi<\infty$, and all $\xi'<\xi$. We notice that if $\xi' \geq 0$ then $x_n \leq \xi'$ if and only if $\hat{\xi}_{n-1} \leq \xi'$. On the other hand, if $\xi'<0$, $P_\xi\{x_n \leq \xi'\}=0$. Therefore,

$$P_\xi(\hat{x}_n \leq \xi') \geq P_\xi(x_n \leq \xi'), \qquad \text{for any } 0 \leq \xi<\infty, \qquad \text{and } \xi'<\xi. \tag{3.6}$$

Let $\delta=\xi-\xi'$.

It follows from (3.6) that

$$P_\xi[\xi-\hat{x}_n \geq \delta] \geq P_\xi[\xi-x_n \geq \delta], \qquad \text{for all } \delta>0. \tag{3.7}$$

In analogy to R_n, define $\hat{R}_n=(\xi-\hat{x}_n)^+$. Inequality (3.7) is equivalent to

$$P_\xi[R_n \geq \delta] \leq P_\xi[\hat{R}_n \geq \delta], \qquad \text{for all } \delta>0. \tag{3.8}$$

Since both R_n and $\hat{R}_n$ are nonnegative random variables we infer that for every $n=1, 2, \ldots$

$$E_\xi[R_n] \leq E_\xi[\hat{R}_n]. \tag{3.9}$$

(Q.E.D.)

4. SEARCH PROCEDURES FOR CASES WITH UNKNOWN VARIANCE

In the present section we consider cases in which a and b are known and σ^2 is unknown. We assume that $\sigma \leq \bar{\sigma}$, where $\bar{\sigma}$ is a given constant. Although $x=0$ does not mean here that we do not administer any drug, there is an "absolute zero" dosage level under which we cannot go. Therefore $\bar{\sigma}$ has to be taken in such a way, that if $\sigma=\bar{\sigma}$ we shall still get ξ_γ larger than this absolute zero dosage. It will be simpler to allow here also negative values for x. In these cases we naturally search for ξ_γ as given by (2.1).

4.1. Model II

We start with Model II for which ξ_γ as in (2.1) assumes the following form:

$$\xi_\gamma=\xi-C\sigma, \tag{4.1}$$

where $\xi=-a/b$ and $C=Z_\gamma/b$. Both ξ and C are known.

The search procedure which is proposed for this case, and which is designated by $P^*.2.$ is specified in the following manner:

Let $x_1=\xi-C\bar{\sigma}$. For each $i=1, 2, \ldots$ let $U_i=Y_i-bx_i-a$, and

$$S_n^2=\sum_{i=1}^{n} U_i^2.$$

Furthermore, let $\sigma_{n,\alpha}^2=\text{Min}\,(S_n^2/\chi_{n,\alpha}^2, \bar{\sigma}^2)$, where $\chi_{n,\alpha}^2$ is the αth fractile of the chi-square distribution with n degrees of freedom.

The dosage for the $(n+1)$st trial is specified by $P^*.2.$ as

$$x_{n+1}=\xi-C\sigma_{n,\alpha}, \qquad n=1, 2, \ldots \tag{4.2}$$

THEOREM 4: *The procedure* P*.2. *is feasible, strongly consistent and optimal*

PROOF: $U_i (i=1, 2, \ldots, n)$ are i.i.d. normal random variables with mean 0 and variance σ^2. Therefore S_n^2/σ^2 has a chi-square distribution with n degrees of freedom. Hence

$$P[S_n^2/\sigma^2 \leq \chi^2_{n,\alpha}] = P[\sigma^2 \geq S_n/\chi^2_{n,\alpha}] = \alpha, \tag{4.3}$$

and $\sigma^2_{n,\alpha}$ is a U.M.A. upper confidence limit for σ^2, at level $1-\alpha$. This implies the feasibility of the procedure. To show strong consistency it is enough to show that $\sigma^2_{n,\alpha} N \xrightarrow{\text{a.s.}} \sigma^2$.

Let us write

$$S_n^2/\chi^2_{n,\alpha} = \left(\sum_{i=1}^{n} U_i^2/n\right)(n/\chi^2_{n,\alpha}).$$

According to the Strong Law of Large Numbers and the Central Limit Theorem we conclude that:

$$\sum_{i=1}^{n} U_i^2/n \xrightarrow{\text{a.s.}} \sigma^2 \text{ as } n \to \infty, \qquad \chi^2_{n,\alpha} \approx n + Z_\alpha \sqrt{2n} \text{ as } n \to \infty.$$

Hence, $\chi^2_{n,\alpha}/n \to 1$. As also $\sigma^2 < \bar{\sigma}^2$ we obtain $\sigma^2_{n,\alpha} \xrightarrow{\text{a.s.}} \sigma^2$.

As $\sigma^2_{n,\alpha}$ is a U.M.A. upper confidence limit for σ^2, at level $1-\alpha$, it follows that $x_{n+1} = \xi - C\sigma_{n,\alpha}$ is also a U.M.A. lower confidence limit for ξ_γ, $(\xi_\gamma = \xi - C\sigma)$ at level $1-\alpha$, for $n=1, 2, \ldots$. The optimality of $P^*2.$ follows then, by the same arguments as in the proof of Theorem 3. (Q.E.D.)

4.2. Model I

A search procedure having similar properties can be found for Model I just by a few modifications. Thus for Model I we set

$$U_i = (Y_i - bx_i - a)/x_i \qquad i=1, 2, \ldots.$$

$U_i (i=1, 2, \ldots, n)$ are again i.i.d. normal random variables with mean 0 and variance σ^2. According to (2.1) $\xi_\gamma = -a/(b+z_\gamma\sigma)$. We employ also here the statistic $\sigma^2_{n,\alpha} = S_n^2/\chi^2_{n,\alpha}$ which is a U.M.A. upper confidence limit at a level $(1-\alpha)$. We thus define the search procedure $P.2.$ for Model I by specifying the dosages:

$$x_1 = -a/(b+z_\gamma\bar{\sigma}),$$

$$x_{n+1} = -a/(b+z_\gamma\sigma^*_{n,\alpha}), \qquad n=1, 2, \ldots, \tag{4.4}$$

in which $\sigma^*_{n,\alpha} = \text{Min}\,(\sigma_{n,\alpha}, \bar{\sigma})$.

THEOREM 5: *The procedure* P.2. *given by* (4.4) *is feasible, strongly consistent and optimal.*

PROOF: By the previous arguments, $\sigma^2_{n,\alpha} \xrightarrow{\text{a.s.}} \sigma^2$ and $x_n \xrightarrow{\text{a.s.}} \xi_\gamma$ as $n \to \infty$. Again the fact that $\sigma^{*2}_{n,\alpha}$ is a U.M.A. upper confidence limit at level $1-\alpha$ for σ^2, implies that $x_{n+1} = -a/(b+z_\gamma\sigma^*_{n,\alpha})$ is a

U.M.A. lower confidence limit for $\xi_\gamma = -a/(b+z_\gamma\sigma)$, at level $1-\alpha$. This implies the optimality of the search procedure $P.2.$ in the same manner as for procedure $P^*.2.$ and $P^*.1.$

5. CASES WITH UNKNOWN INTERCEPT AND VARIANCE

We assume here that the slope b is known, but the intercept a and the variance σ^2 are unknown. We have a solution only for Model II, where σ^2 is fixed. The search is for ξ_γ which is specified in (2.1), and is subjected to the further condition that $\xi_\gamma \geq 0$. Since here we need at least two observations in order to estimate σ, take the first two dosages at $x_1 = x_2 = 0$. Let $U_i = Y_i - bx_i \quad i = 1, 2, \ldots$). Also here the U_i's are i.i.d. random variables having a normal distribution $N(a, \sigma^2)$. Let

$$\bar{U}_n = \sum_{i=1}^{n} U_i/n$$

and

$$S_n^2 = \sum_{i=1}^{n} (U_i - \bar{U}_n)^2/(n-1)$$

for $n = 2, 3, \ldots$.

Notice that the expectation of $b\xi_\gamma + \bar{U}_n$ is $b\xi_\gamma + a = -Z_\gamma\sigma$. Hence $\sqrt{n}\ (b\xi_\gamma + \bar{U}_n)/S_n$ has a noncentral t distribution with $n-1$ degrees of freedom and noncentrality parameter $-\sqrt{n}Z_\gamma$. Let $t_\alpha[n-1, -\sqrt{n}Z_\gamma]$ denote the α-fractile of this noncentral t distribution. Then,

$$P_{a,\sigma}[\sqrt{n}\ (b\xi_\gamma + \bar{U}_n)/S_n \geq t_\alpha[n-1, -\sqrt{n}Z_\gamma]] = 1-\alpha, \qquad \text{for all } a, \sigma. \tag{5.1}$$

Thus,

$$\hat{\xi}_{n,\alpha} = S_n t_\alpha[n-1, -\sqrt{n}\ Z_\gamma]/\sqrt{n}\ b - \bar{U}_n/b \tag{5.2}$$

is a U.M.A. invariant $1=\alpha$ lower confidence limit for ξ_γ. If $X_{n+1} = \text{Max}\ (0, \hat{\xi}_{n,\alpha})$ for $n = 2, 3, \ldots$ we have a feasible procedure.

To prove strong consistency we have to show that $\hat{\xi}_{n,\alpha} \xrightarrow{\text{a.s.}} \xi_\gamma$. This follows immediately from the fact that $\bar{U}_n \xrightarrow{\text{a.s.}} a$ and $S_n \xrightarrow{\text{a.s.}} \sigma$, and that $(t_\alpha[n-1, -\sqrt{n}Z_\gamma]/\sqrt{n} - Z_\alpha/\sqrt{n} + Z_\gamma) \to 0$ as $n \to \infty$. Indeed $t_\alpha[n-1, -\sqrt{n}\ Z_\gamma]$ is asymptotically like the αth fractile of the normal distribution with mean $-Z_\gamma\sqrt{n}$ (the noncentrality parameter) and variance 1. Define now the sequential search procedure $P^*.3.$ Accordingly let $x_1 = x_2 = 0$ and $x_{n+1} = \text{Max}\ (0, \hat{\xi}_{n,\alpha})$ for all $n = 2, 3, \ldots$.

The same arguments as used in the proof of Theorem 3 imply that the procedure $P^*.3.$ minimizes the value $\sum_{i=1}^{N} E(\xi_\gamma - x_i)^+$ for every fixed integer N, among all feasible procedures, which are invariant under linear transformations on the values of Y_i. We shall call this property invariant optimal. We have then proven:

THEOREM 6: *The sequential procedure* P*.3. *is feasible, strongly consistent, and invariant optimal.*

6. CASES OF UNKNOWN SLOPE AND VARIANCE

In the present section we provide a solution only for Model I. Here we define the auxiliary variables $U_1, U_2, \ldots$ in the form

$$U_i = (Y_i - a)/x_i, \qquad i = 1, \ldots . \tag{6.1}$$

Then U_i, $i = 1, 2, \ldots$ are i.i.d. random variables normally distributed with expectation b and variance σ^2. $\bar{U}_n$ and S_n^2 are as before the sample mean and the sample variance based on the first n U_i's. Routine arguments concerning the distributions of these statistics yield immediately that $\sqrt{n}\,(\bar{U}_n \xi_\gamma + a)/\xi_\gamma S_n$ has a noncentral t distribution with $n-1$ degrees of freedom and noncentrality parameters $-Z_\gamma \sqrt{n}$.

In order to obtain a feasible procedure we assume the knowledge of a value x^* such that $0 < x^* < \xi_\gamma$.

Define the procedure $P.4.$ in the following manner:

Let $x_1 = x_2 = x^*$, and

$$x_{n+1} = \text{Max}\,(x^*, \xi_{n-\alpha}), \qquad n = 2, 3, \ldots \tag{6.2}$$

where

$$\hat{\xi}_{n,\alpha} = -a/(\bar{U}_n = S_n t_\alpha[n-1, -Z_\gamma \sqrt{n}]/\sqrt{n}). \tag{6.3}$$

THEOREM 7: *The sequential procedure* P.4. *is feasible, strongly consistent, and invariant optimal.*

PROOF: From the definition of $t_\alpha[n-1, -Z_\gamma \sqrt{n}]$ we obtain that

$$P_{b,\sigma}\{\xi_\gamma \geq a/\bar{U}_n - S_n t_\alpha[n-1, -Z_\gamma \sqrt{n}]/\sqrt{n})\} = 1-\alpha. \tag{6.4}$$

Hence $P\{\xi_\gamma \geq \hat{\xi}_{n,\alpha}\} = 1-\alpha$. This proves feasibility. For consistency we notice that $\bar{U}_n \to b$ a.s., $S_n \to \sigma$ a.s. and $t_\alpha[n-1, -Z_\gamma \sqrt{n}]/\sqrt{n} \to -Z_\gamma$ as $n \to \infty$. Hence, according to (6.2)–(6.3) $x_n \to \xi_\gamma$ a.s. Finally, the invariant optimality follows exactly in the same way as for $P^*.3$. (Q.E.D.)

REFERENCES

[1] Eichhorn, B. H. and S. Zacks, "Sequential Search Of An Optimal Dosage I," submitted for publication to the Journal of American Statistical Association (1972).

[2] Lehmann, E., *Testing Statistical Hypotheses* (John Wiley & Sons, New York, 1959).

FURTHER LIGHT ON NONPARAMETRIC SELECTION EFFICIENCY

Edward J. Dudewicz*
The Ohio State University

and

Chung-lien Fan
The University of Rochester

ABSTRACT

A nonparametric selection procedure $\mathscr{P}_{BS}$ was proposed by Bechhofer and Sobel (1958) and studied by Dudewicz (1971) in comparison with other procedures under normal and uniform alternatives. He found $\mathscr{P}_{BS}$ always required larger sample sizes, sometimes substantially so. For 2-point populations we find more extreme results. We also find that $\mathscr{P}_{BS}$ may be substantially better than reasonable competitors designed specifically for 2-point populations. Finally, a new nonparametric selection procedure (conjectured to be better than $\mathscr{P}_{BS}$) is proposed.

1. INTRODUCTION

Let $\pi_1, \pi_2, \ldots, \pi_k$ be $k(\geq 2)$ populations such that if an observation X is drawn from π_i then

$$P[X = c_i] = 1 - p,\ P[X = s + c_i] = p \qquad (i = 1, \ldots, k), \tag{1.1}$$

where $s(s > 0)$ and $p(0 < p < 1)$ are known and $c_1, \ldots, c_k$ are unknown. We assume that the association between $\pi_1, \ldots, \pi_k$ and $c_{[1]}, \ldots, c_{[k]}$ (where $c_{[1]} \leq \ldots \leq c_{[k]}$ denote the ordered $c_1, \ldots, c_k$) is completely unknown, and that the *best* population is that with the largest location parameter $c_{[k]}$.

In this paper we consider the problem of selecting the population associated with $c_{[k]} = \max(c_1, \ldots, c_k)$. The procedures we will consider each take n independent observations per population in a single stage, where n is set so that the probability of a *correct selection* (CS) satisfies

$$P(CS) \geq P^* \text{ whenever } c_{[k]} - c_{[k-1]} \geq \delta^* \tag{1.2}$$

where $P^*(1/k < P^* < 1)$ and $\delta^*(\delta^* > 0)$ are specified in advance by the experimenter.

In Section 2 we consider a procedure based on sample means, say $\mathscr{P}_M$. Let $\bar{X}_i$ denote the sample mean of the n observations from $\pi_i (1 \leq i \leq k)$, and denote the ordered $\bar{X}_1, \ldots, \bar{X}_k$ by $\bar{X}_{[1]} \leq \ldots \leq \bar{X}_{[k]}$. $\mathscr{P}_M$ selects that population which yielded $\bar{X}_{[k]}$ (the largest sample mean). Note that (for $i \neq j$) $P[\bar{X}_i = \bar{X}_j] = 0$ unless for some $0 \leq l, m \leq n$ we have

$$nc_i + ls = nc_j + ms$$

*This author's research was supported by ONR contract N00014-68A-0091 and by the U.S. Army Research Office–Durham.

(1.3) $$(l-m)s = n(c_j - c_i)$$

$$l - m = n(c_j - c_i)/s.$$

Since $l-m$ is an integer, (1.3) holds iff $s \mid n(c_j - c_i)$, which is false for almost all real s.

In Section 3 we consider a procedure specifically designed for two-point populations, say $\mathscr{P}_S$. This procedure is intuitively appealing (and, for two-point populations, may be optimal among all single-stage procedures which take n observations from each population). Some applications, the most important of which deals with the problem of the repair (or replacement) of the worst of a group of machines, are given; however, the main values of our study of ρ_S are probably the light shed on the efficiency of a certain nonparametric selection procedure (see Section 4) and the expository value.

In Section 4 we use $\mathscr{P}_M$ and $\mathscr{P}_S$ to shed further light on the efficiency of a nonparametric selection procedure suggested by Bechhofer and Sobel (1958) and studied by Dudewicz (1971).

In Section 5 we propose a new nonparametric selection procedure, which we hope to study in detail in a later paper.

2. THE MEANS PROCEDURE

Recall that we take n independent observations per population in a single stage. For the means procedure $\mathscr{P}_M$ (see Section 1), the $P(CS)$ is minimized (over $c_{[k]} - c_{[k-1]} \geq \delta^*$) when $c_{[1]} = \cdots = c_{[k-1]} = c_{[k]} - \delta^*$, and is then

(2.1) $$P_L(CS) = \sum_{j=0}^{n} \left(F\left(j + n\frac{\delta^*}{s}\right)\right)^{k-1} p\left(c_{[k]} + \frac{j}{n}s\right),$$

where

(2.2) $$p\left(c_{[k]} + \frac{j}{n}s\right) = \binom{n}{j} p^j (1-p)^{n-j} \qquad (j = 0, 1, \ldots, n),$$

and where

(2.3) $$F(x) = \begin{cases} 1 & \text{if } x > n \\ \sum_{i=0}^{[x]} \binom{n}{i} p^i (1-p)^{n-i} & \text{if } 0 \leq x \leq n \\ 0 & \text{if } x < 0. \end{cases}$$

Note that $P_L(CS) = 1$ if $\delta^* > s$ (iff (say) $\lambda^* = \delta^*/s > 1$), in which case (1.2) can be trivially satisfied by taking $n = 1$. Hence we now assume $0 < \delta^* < s$ (i.e. $0 < \lambda^* < 1$). (As a check of (2.1) note that $P_L(CS) = 1$ if $p = 0$ or $p = 1$.)

If p were unknown, one might desire to set n so that

(2.4) $$\inf_{0 \leq p \leq 1} P_L(CS) = \inf_{0 \leq p \leq 1} \sum_{j=0}^{n} \left(\sum_{i=0}^{[j+n\lambda^*]} \binom{n}{i} p^i (1-p)^{n-i}\right)^{k-1} \binom{n}{j} p^j (1-p)^{n-j} = P^*.$$

The p at which $P_L(CS)$ is minimized is a function of n, k, λ^* for which we have no general simple expression. When $n = 1$ $\mathscr{P}_M$ is the same as $\mathscr{P}_S$, and an exact solution has been obtained in Section 3. When $k = 2$ it is easily shown directly that $P_L(CS)$ is minimized at $p = 0.5$. (Calculations of $P_L(CS)$ for $p = 0.00(0.01)1.00$ at $\{n = 1(1)5;\ k = 2, 5;\ \lambda^* = 0.1^+(0.1^+)0.9^+\}$ show that the minimizing p is between 0.33 and 0.50, and that $P_L(CS)$ at the inf differs little from $P_L(CS)$ at $p = 0.5$.)

An interesting phenomenon showed up in our claculations: $P_L(CS)$ at fixed p, k, λ^* does not necessarily increase as n increases for small n. (E.g. for $k = 2$, $p = 0.5$, $\lambda^* = 0.1$, $(n, P_L(CS))$ takes on the values (1, 0.75), (2, 0.69), (3, 0.66), (4, 0.64), (5, 0.62). Note that $n\lambda^*$ is not an integer in any of these cases; as we say at (1.3), this is necessary in order that ties have probability zero.)

3. THE TWO-POINT PROCEDURE

We now wish to specify the two-point procedure $\mathscr{P}_S$ precisely. First we take n independent observations per population in a single stage. These allow as to conceive of decomposing

$$\Pi = \{\pi_1, \ldots, \pi_k\} \tag{3.1}$$

into three disjoint (random) sets with union Π, namely*

$$\mathscr{L} = \left\{\pi : \pi \epsilon \Pi \text{ and the sample from } \pi \text{ yielded its lower value } n \text{ times}\right\} \tag{3.2a}$$

$$\mathscr{U} = \left\{\pi : \pi \epsilon \Pi \text{ and the sample from } \pi \text{ yielded its upper value } n \text{ times}\right\} \tag{3.2b}$$

$$\mathscr{T} = \left\{\pi : \pi \epsilon \Pi \text{ and the sample from } \pi \text{ yielded both its upper and lower values at least once each}\right\}. \tag{3.2c}$$

We then select that π which yielded the single** largest value.

We now wish to find the $P(CS)$ for procedure $\mathscr{P}_S$. Let $\pi_{(k)}$ denote the best population, and let $\nu + 1$ denote the number of $c_i + s$ which exceed $c_{[k]}$. Then

$$\begin{aligned} P(CS) &= P(CS \mid \pi_{(k)} \epsilon \mathscr{L}) P(\pi_{(k)} \epsilon \mathscr{L}) + P(CS \mid \pi_{(k)} \epsilon \mathscr{U}) P(\pi_{(k)} \epsilon \mathscr{U}) + P(CS \mid \pi_{(k)} \epsilon \mathscr{T}) P(\pi_{(k)} \epsilon \mathscr{T}) \\ &= ((1-p)^n)^\nu (1-p)^n + 1 \cdot p^n + 1 \cdot (1 - p^n - (1-p)^n) \\ &= 1 - (1-p)^n + ((1-p)^n)^{\nu+1}. \end{aligned} \tag{3.3}$$

*In practice we cannot distinguish among members of $\mathscr{L} \cup \mathscr{U}$.

**Reasoning as at (1.3), ties can occur if and only if:

(1) $c_{[k-1]} + s = c_{[k]} + s$, or (2) $c_{[k-1]} + s = c_{[k]}$, or (3) more than two $c_i + s$ exceed $c_{[k]}$ and at least two of these are equal. Since if $c_{[k-1]} + s = c_{[k]} + s$ we have two "best" populations, it is easy but tedious to verify that (even if ties occur and we randomize over tied populations) equation (3.3) is still valid in cases (1) and (2), while in case (3) the $P(CS)$ is increased slightly. We exclude case (3), considering such configurations only as limits (in which case we consider the limits of probabilities as this configuration is approached, and not the probabilities in the limiting configuration).

This $P(CS)$ is minimized (over $c_{[k]} - c_{[k-1]} \geq \delta^*$) when $c_{[1]} = \ldots = c_{[k-1]} = c_{[k]} - \delta^*$ (i.e. when ν is maximized to $\nu = k - 1$) and is then (assuming $0 < \delta^* < s$)

$$P_L(CS) = 1 - (1-p)^n + (1-p)^{nk}. \tag{3.4}$$

In many cases one will not know p, and may desire to set n so that

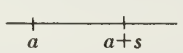
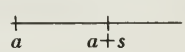
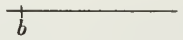
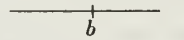
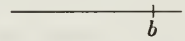
$$\inf_{0 \leq p \leq 1} P_L(CS) = P^*. \tag{3.5}$$

The $P_L(CS)$ is minimized at $p = 1 - \left(\frac{1}{k}\right)^{\frac{1}{(k-1)n}}$. (The following table illustrates typical values.) Note that at the minimizing p the probability is not dependent upon n and (3.5) cannot in general be achieved when p is unknown:

$$\inf_{0 \leq p \leq 1} P_L(CS) = 1 - (1/k)^{\frac{1}{k-1}} + (1/k)^{\frac{k}{k-1}}. \tag{3.6}$$

	$1 - (1/k)^{\frac{1}{(k-1)n}}$										inf $P_L(CS)$
k \ n	1	2	3	4	5	6	7	8	9	10	Any
2	.50	.29	.21	.16	.13	.11	.09	.08	.07	.07	.75
3	.42	.24	.17	.13	.10	.09	.08	.07	.06	.05	.62
4	.37	.21	.14	.11	.09	.07	.06	.06	.05	.05	.53
5	.33	.18	.13	.10	.08	.06	.06	.05	.04	.04	.47
6	.30	.16	.11	.09	.07	.06	.05	.04	.04	.04	.42
7	.28	.15	.10	.08	.06	.05	.05	.04	.04	.03	.38
8	.26	.14	.09	.07	.06	.05	.04	.04	.03	.03	.35
9	.24	.13	.09	.07	.05	.04	.04	.03	.03	.03	.32
10	.23	.12	.08	.06	.05	.04	.04	.03	.03	.03	.30

Remark. Note that if $\pi_{(k)} \in \mathcal{T} \cup \mathcal{U}$ we always make a correct selection. However when $\pi_{(k)} \in \mathcal{L}$ we won't make a correct selection unless all populations π_i such that $c_i + s > c_{[k]}$ are also in $\mathcal{L}$. One might try to modify $\mathcal{P}_S$ in order to increase the chances of a correct selection when $\pi_{(k)} \in \mathcal{L}$, reasoning as follows. Any of three configurations of "best in $\mathcal{T}$" *vs.* "best in $\mathcal{L} \cup \mathcal{U}$" can occur:

"Best in $\mathcal{T}$"			
"Best in $\mathcal{L} \cup \mathcal{U}$"			
Case	(i)	(ii)	(iii)

In situation (i) ((iii)) we select the population yielding $(a, a+s)$ $((b))$, and are certain of having selected correctly. In situation (ii) $\pi_{(k)}$ could be either in $\mathscr{T}$ or in $\mathscr{L} \cup \mathscr{U}$. $\mathscr{P}_S$ always selects the best in $\mathscr{T}$ in this case. One might also study rules which select the population yielding (b) in case (ii) (e.g. so select when $p < c_0$). We have not chosen to do so since (1) such a rule does not uniformly increase the chances of a correct selection, (2) such a rule is not as simple and intuitively appealing, and (3) such a rule is dependent upon knowledge of p (whereas $\mathscr{P}_S$ can be utilized regardless of the value of p . . . although of course its characteristics depend upon the value of p).

Applications

Suppose we are considering k machines with the following characteristics. The output of each machine is either "a" or "$a+s$" with respective probabilities of "$1-p$" and "p" (two modes of operation). The probability "p" may be known or unknown, but is the same for all machines (same modal probabilities). However, the value of "a" may be different for different machines. We may wish to select, after observing some outputs, that machine with the largest "a" (the one that has "slipped" the most) and repair or replace it (repair or replacement of the worst of a group of machines).

As a second application (this one hypothetical) consider mining. Suppose that a lode (location unknown) results in precious stones being scattered at sites in an area of 10 (say) square miles. Each site has stones of low grade "a" or high grade "$a+s$", in proportions $1-p$ and p respectively, "s" is fixed, "p" is unknown, and "a" varies from site to site. Based on samples of scattered stones, select the area of highest potential value for intensive search.

Note that the two-point distribution has been used extensively by Hald and Thyregod [4] as a prior distribution for Bayesian sampling plans. The parameters of the prior are usually estimated from previously-inspected lots (see pp. 36–37 of Hald and Thyregod [4]), with each such lot being assumed to have the same probability distribution. When lots come from several sources, considerations related to those of the present paper may be important, but have not yet been explored.

4. THE BECHHOFER-SOBEL NONPARAMETRIC PROCEDURE

The nonparametric procedure $\mathscr{P}_{BS}$ proposed by Bechhofer and Sobel (1958) takes n independent vectors of observations. Then the largest observation in each vector is assigned a 1, and all other observations are assigned 0. (Ties are broken at random in the assignment.) The population with the most 1's is selected as best (ties being broken at random). Let p_i be the probability that an observation from π_i exceeds observations from $\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_k$ $(1 \leq i \leq k)$ (assuming, without loss, that ties have probability zero). Let $p_{[1]} \leq \ldots \leq p_{[k]}$ denote the ranked $p_1, \ldots, p_k$. Then we have essentially the problem of choosing the sample size n for the problem of selecting the cell with the highest probability in a multinomial distribution. To attain $P(CS) \geq P^*$ whenever $p_{[k]} \geq \theta^* p_{[k-1]}$ we can use the tables of Bechhofer, Elmaghraby and Morse [1], calculated under the least favorable configuration

$$p_{[1]} = \ldots = p_{[k-1]} = p_{[k]}/\theta^*. \tag{4.1}$$

For fixed $\lambda^*(0 < \lambda^* < 1)$ and k, we can find p or $p_{[k]}$ for our two-point populations from the relation

$$p_{[k]} = p + (1-p)^k. \tag{4.2}$$

If we select k and θ^*, then $p_{[k]}$ is determined as $\theta^*/(\theta^*+k-1)$ and, via (4.2), p is found. Note that (4.2) will have either no, one, or two solutions (depending on θ^* and k).

The following table illustrates the relative efficiency of $\mathscr{P}_M$ *vs.* $\mathscr{P}_{BS}$ for various P^*, k, θ^*, λ^*. The *RE* is the sample size needed by $\mathscr{P}_M$ divided by the sample size needed by $\mathscr{P}_{BS}$. (Note that two values per cell can arise when (4.2) has two solutions for p. All *RE*'s are ratios of the smallest integer sample sizes which guarantee the probability requirement.)

RE = (*Sample Size for* $\mathscr{P}_M$)/(*Sample Size for* $\mathscr{P}_{BS}$)

	k	2								
θ^*	P^* \ λ^*	0.1+	0.2+	0.3+	0.4+	0.5+	0.6+	0.7+	0.8+	0.9+
3.00	0.99		3.42	1.42	0.79	0.53	0.37	0.32	0.21	0.21
	0.90	11.43	2.86	1.00	0.71	0.29	0.29	0.29	0.29	0.29
	0.80	10.00	1.67	1.33	1.00	0.67	0.67	0.67	0.67	0.67
10.00	0.9995			1.82	1.18	0.73	0.45	0.45	0.36	0.36
	0.999		4.44	2.22	1.11	0.67	0.56	0.33	0.33	0.33
	0.99		4.00	2.00	1.00	0.40	0.40	0.40	0.40	0.40
	0.90	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	0.80	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

	k	3			4		
θ^*	P^* \ λ^*	0.3+	0.6+	0.9+	0.3+	0.6+	0.9+
10.00	0.9995	1.85, 2.85	0.38, 0.69	0.31, 0.38	1.60, 3.33	0.47, 0.80	0.27, 0.33
	0.999	1.82, 3.09	0.45, 0.82	0.36, 0.36	1.54, 3.38	0.38, 0.77	0.31, 0.38
	0.99	1.43, 2.86	0.57, 0.71	0.43, 0.43	1.56, 3.00	0.44, 1.25	0.33, 0.44
	0.90	1.33, 2.33	0.67, 0.67	0.67, 0.67	1.00, 2.50	0.50, 0.50	0.50, 0.50
	0.80	1.00, 1.00	1.00, 1.00	1.00, 1.00	1.33, 1.33	0.67, 0.67	0.67, 0.67

The following table illustrates the *RE* of $\mathscr{P}_S$ *vs.* $\mathscr{P}_{BS}$ for various P^*, k, θ^*. (For $\theta^* = 3.00$, $k = 2$ only the values 0.36, 0.57, 0.67 are available, for $P^* = 0.99, 0.90, 0.80$ respectively.)

RE = (*Sample Size for* $\mathscr{P}_S$)/(*Sample Size for* $\mathscr{P}_{BS}$)

$\theta^* = 10.00$			
P^* \ k	2	3	4
0.9995	0.45, 6.55	0.38, 5.77	0.40, 5.20
0.999	0.44, 7.22	0.36, 6.18	0.38, 5.46
0.99	0.60, 8.80	0.43, 6.57	0.44, 5.33
0.90		0.67, 7.67	0.50, 6.00
0.80			0.67, 5.67

Under normal and uniform alternatives, Dudewicz (1971) found efficiencies for $\mathscr{P}_{BS}$ *vs.* $\mathscr{P}_M$, and for $\mathscr{P}_{BS}$ *vs.* procedures specially designed for uniform or normal populations. These were between 0.27 and 0.74 (i.e. in those cases the nonparametric procedure $\mathscr{P}_{BS}$ always required from 30% to 300% large samples). We have found efficiencies for $\mathscr{P}_{BS}$, in the case of 2-point populations, which range from 0.21 to 11.43. This shows that (as was conjectured) 2-point populations furnish a more extreme alternative, and that for some populations the nonparametric procedure $\mathscr{P}_{BS}$ can be much more efficient than reasonable parametric alternatives.

5. A NEW NONPARAMETRIC PROCEDURE

The nonparametric selection procedure $\mathscr{P}_{BS}$ of Bechhofer and Sobel [2] took n vectors of observations $(X_{ij}, \ldots, X_{kj})$ $(j = 1, 2, \ldots, n)$. In each vector the largest observation was assigned a 1 (the others 0's), and the population which amassed the most 1's was selected. Tables which could be used to implement $\mathscr{P}_{BS}$ were given by Bechhofer, Elmaghraby and Morse [1]. Dudewicz [3] showed that $\mathscr{P}_{BS}$ has reasonable efficiency relative to certain alternatives, and (in Section 4 of this paper) the present authors found results indicating that $\mathscr{P}_{BS}$ may be surprisingly good in certain instances.

Now, replacing $(X_{1j}, \ldots, X_{kj})$ by $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ (the 1 replacing the largest of $X_{1j}, \ldots, X_{kj}$) intuitively seems a great loss of information. If instead one replaced the largest observation by k, the next largest by $k-1$, . . ., the smallest by 1, and selected that population achieving the largest rank sum, how would one do? It seems intuitively clear to us that this procedure should be better than $\mathscr{P}_{BS}$ (at least under mild restrictions). It is not clear that this new procedure ($\mathscr{P}_N$, say) has an easily-determined least favorable configuration; $\mathscr{P}_{BS}$ did. (Considering the observations vector be vector helps avoid least favorable configuration problems found by Rizvi and Woodworth [7]. These difficulties invalidated most then-existing nonparametric selection procedures except $\mathscr{P}_{BS}$ and some "subset-selection" procedures.) The use of $\mathscr{P}_N$ for subset selection has recently been studied by McDonald [5, 6]. The present authors and Dr. Gary C. McDonald are now studying $\mathscr{P}_N$ in detail in terms of efficiency, tables needed for implementation, comparisons with other nonparametric procedures, and modifications for other problems, and hope to present these considerations in a later paper.

REFERENCES

[1] Bechhofer, R. E., S. Elmaghraby, and N. Morse, "A single-sample multiple-decision procedure for selecting the multinomial event which has the highest probability," Annals of Mathematical Statistics, Vol. 30, pp. 102–119 (1959).

[2] Bechhofer, R. E. and M. Sobel, "Non-parametric multiple-decision procedures for selecting that one of k populations which has the highest probability of yielding the largest observation," Abstract, Annals of Mathematical Statistics, Vol. 29, p. 325 (1958).

[3] Dudewicz, E. J. "A nonparametric selection procedure's efficiency: largest location parameter case," Journal of the American Statistical Association, Vol. 66, pp. 152–161 (1971).

[4] Hald, A. and P. Thyregod, "Bayesian single sampling plans based on linear costs and the Poisson distribution," Matematiskfysiske Skrifter udgivet af Det Kongelige Danske Videnskabernes Selskab, Vol. 3, No. 7, pp. 1–100 (1971).

[5] McDonald, G. C. "Some multiple comparison selection procedures based on ranks," Sankhyā, Series A, Vol. 34, pp. 53–64 (1972).

[6] McDonald, G. C. "The distribution of some rank statistics with applications in block design selection problems," *Research Publication GMR–1209*, Research Laboratories, General Motors Corporation, Warren, Michigan (1972). To appear, Sankhya.

[7] Rizvi, M. H. and G. G. Woodworth, "On selection procedures based on ranks: counterexamples concerning least favorable configurations," Annals of Mathematical Statistics, Vol. 41, pp. 1942–1951 (1970).

SIMPLIFIED ESTIMATES OF THE PARAMETERS OF THE DOUBLE EXPONENTIAL DISTRIBUTION BASED ON OPTIMUM ORDER STATISTICS FROM A MIDDLE-CENSORED SAMPLE

M. Ahsanullah
Food and Drug Directorate, Ottawa

and

M. A. Rahim*
Statistics Division
Indian and Northern Affairs
Ottawa, Canada

ABSTRACT

In an ordered sample from a given population, a few of the consecutive observations from somewhere in the middle may be missing. Further, we may be constrained to use a few, and not all, of the remaining observations for purposes of estimation of population parameters. In this paper, such a situation is considered for the double exponential distribution and best linear unbiased estimates are obtained for its parameters, based on a choice of an optimum set of order statistics when the number of observations in the set are prefixed.

1. INTRODUCTION

The density function of the double exponential distribution is given by

$$f_x(x, \theta_1, \theta_2,) = \frac{1}{2\theta_2} e^{-\frac{|x-\theta_1|}{\theta_2}}, \qquad -\infty < x < \infty \quad 0 < \theta_2. \tag{1.1}$$

The problem of estimating the parameters θ_1 and θ_2, based on order statistics, have been studied in the past under various situations.

Sarhan [4] gave the best linear unbiased estimate (BLUE) of θ_1 and θ_2 based on order statistics when the original sample was of size $n=5$ and the available sample, after censoring, was $n'=2(1)5$. He considered left or right censored samples and used all the n' observations. Govindarajulu [2] extended these results for $n=20$ and $n'=2(1)20$ when censoring was done symmetrically from left and right. He also used all the n' observations. Chan and Chan [1] obtained BLUE of θ_1 and θ_2 based on a choice of $k(\leq n)$ optimum order statistics for $n=1(1)20$ and $k(\leq n)=1(1)4$. He considered the case when full sample n was available.

There are, however, practical situations, when the censoring occurs from somewhere in the middle of an ordered sample. Sarhan and Greenberg [5] mentioned the case of telemetry where signals are sent at regular intervals and a few may be missing during the journey. Middle-censored samples may also occur due to failure of the measuring instrument to record observations or due to off-shifts

*This research was performed when the author was at Carleton University, Ottawa, Canada.

or weekend interruptions during the course of an experiment—particularly when the variate under observation is a time period (i.e., the period to failure of a piece of equipment undergoing testing; survival period of bacteria, etc.).

From such a middle censored sample, one may further decide—for reasons of economy or practical convenience—to use a few, and not all of the available observations. The question then arises, which of the available observations to use?

In this paper we consider such a situation when we have an ordered sample of size n,

$$x_{(1)} < x_{(2)} < \ldots < x_{(R_1)} < x_{(R_2)} < x_{(R_2+1)} < \ldots < x_{(n)},$$

where R_1 and R_2 are some integers such that $1 \leq R_1 < R_2 \leq n$ and the middle observations $x_{(R_1+1)}$, $x_{(R_1+2)}, \ldots x_{(R_2-1)}$, are missing. For a given $k = k_1 + k_2$, such that $k_1 < R_1$ and $k_2 < n - (R_2 - 1)$, our objective is to determine the optimum ranks $n_1^0, n_2^0, \ldots n_{k_1}^0, n_{k_1+1}^0, \ldots n_{k_1+k_2}^0$, such that

$$1 \leq n_1^0 < n_2^0 < \ldots \ldots < n_{k_1}^0 \leq R_1$$

$$R_2 \leq n_{(k_1+1)}^0 < n_{(k_1+2)}^0 < \ldots . < n_{(k_1+k_2)}^0 \leq n$$

and obtain BLUE of θ_1 and θ_2 based on the observations

$$x_{(n_1^0)}, x_{(n_2^0)}, \ldots x_{(n_{k_1}^0)}, x_{(n_{k_1+1}^0)} \ldots x_{(n_{k_1+k_2}^0)}.$$

2. OPTIMUM RANKS AND "BLUE" OF θ_1 AND θ_2

Consider any set of ranks $n_1, n_2, \ldots . n_{k_1}, n_{k_1+1} \ldots n_{k_1+k_2}$ such that

$$1 \leq n_1 < n_2 < \ldots . < n_{k_1} \leq R_1, R_2 \leq n_{k_1+1} < n_{k_1+2} < \ldots < n_{k_1+k_2} \leq n.$$

Then, for the double exponential distribution $(1 \cdot 1)$, transforming to standardized variable z, so that $x = \theta_1 + \theta_2 Z$, we can write

$$E(x_{(n_i)}) = \theta_1 + \theta_2 E(z_{(n_i)}), \quad i = 1, 2 \ldots . k_1, k_1+1, \ldots k_1+k_2 \tag{2.1}$$

$$V(x_{(n_i)}) = \theta_2^2 V(z_{(n_i)})$$

$$\text{Cov}\,(x_{(n_i)}, x_{(n_j)}) = \theta_2^2\,\text{Cov}\,(z_{(n_i)}, z_{(n_j)}).$$

Hence, by applying the generalized least square theorem and following Lloyd [3], the BLUE's of θ_1 and θ_2 can be obtained in three different situations, namely, when $\theta_1 = \theta_1^0$ is known, or $\theta_2 = \theta_2^0$ is known, or θ_1 and θ_2 are both unknown. These best linear unbiased estimates and their variances and generalized variance are shown below, which we have expressed in terms of following matrices

$$Y' = [x_{(n_1)}, x_{(n_2)}, \ldots . x_{(n_{k_1}+n_{k_2})}], \qquad \alpha' = [E(z_{(n_1)}), E(z_{(n_2)}), \ldots E(z_{(n_{k_1}+k_2)})$$

$$1' = [1, 1, \ldots . 1], \qquad V = [\mathrm{Cov}\,(z_{(n_i)}, z_{(n_j)})].$$

CASE 1: $\theta_1 = \theta_1^0$ is known:

The BLUE of θ_2 denoted by $\hat{\theta}_2$ and its variance $V(\hat{\theta}_2)$ are given by

$$\hat{\theta}_2 = \frac{\alpha' V^{-1} Y}{\alpha' V^{-1} \alpha} - \frac{\theta_1^0 (\alpha' V^{-1} 1)}{\alpha' V^{-1} \alpha}, \tag{2.2}$$

$$= b_{(n_1)} x_{(n_1)} + b_{(n_2)} x_{(n_2)} + \ldots + b_{(n_{k_1+k_2})} x_{(n_{k_1+k_2})} - b\theta_1^0;$$

$$V(\hat{\theta}_2) = \frac{\theta_2^2}{(\alpha' V^{-1} \alpha)}. \tag{2.3}$$

CASE 2: $\theta_2 = \theta_2^0$ is known:

The BLUE of θ_1 denoted by $\hat{\theta}_1$ and its variance $V(\hat{\theta}_1)$ are given by

$$\hat{\theta}_1 = \frac{(1' V^{-1} Y)}{(1' V^{-1} 1)} - \frac{\theta_2^0 (1' V^{-1} \alpha)}{(1' V^{-1} 1)} \tag{2.4}$$

$$= a_{(n_1)} x_{(n_1)} + a_{(n_2)} x_{(n_2)} + \ldots . + a_{(n_{k_1+k_2})} x_{(n_{k_1+k_2})} - a\theta_2^0;$$

$$V(\hat{\theta}_1) = \theta_2^2 / (1' V^{-1} 1). \tag{2.5}$$

CASE 3: θ_1, θ_2 both unknown:

The BLUE's of θ_1, θ_2 denoted by $\hat{\theta}_1, \hat{\theta}_2$ and their generalized variance $GV(\hat{\theta}_1\ \hat{\theta}_2)$ are given by

$$\hat{\theta}_1 = -\alpha' \Gamma Y, \Gamma = V^{-1}(1\alpha' - \alpha 1') \frac{V^{-1}}{\Delta}, \Delta = (1' V^{-1} 1)(\alpha' V^{-1} \alpha) - (1' V^{-1} \alpha)^2 \tag{2.6}$$

$$\hat{\theta}_2 = 1' \Gamma Y$$

or

$$\hat{\theta}_1 = a_{(n_1)} x_{(n_1)} + a_{(n_2)} x_{(n_2)} + \ldots . + a_{(n_{k_1+k_2})} x_{(n_{k_1+k_2})}$$

$$\hat{\theta}_2 = b_{(n_1)} x_{(n_1)} + b_{(n_2)} x_{(n_2)} + \ldots . + b_{(n_{k_1+k_2})} x_{(n_{k_1+k_2})}.$$

$$GV(\hat{\theta}_1 \hat{\theta}_2) = V(\hat{\theta}_1) V(\hat{\theta}_2) - (\mathrm{Cov}\ (\hat{\theta}_1, \hat{\theta}_2))^2 \tag{2.7}$$

$$= \theta_2^4 / [(\alpha' V^{-1} \alpha)(1' V^{-1} 1) - (1' V^{-1} \alpha)^2].$$

Now, for a given Y the 'a' and 'b' coefficients in the above expressions for BLUE's are determinable by direct matrix multiplication, when α and V are known, i.e., the quantities $E(z_{(n_i)})$, $V(z_{(n_i)})$, $\mathrm{Cov}(z_{(n_i)}, z_{(n_j)})$ are known. In fact, these quantities have been tabulated by Govindarajulu [2]. Hence, using those values; for any fixed n, truncation points R_1, R_2, and k; it is possible to take all combination of ranks and compute the values of $V(\hat{\theta}_1)$, $V(\hat{\theta}_2)$, or $GV(\hat{\theta}_1\ \hat{\theta}_2)$, as the case may be. We then take that set of ranks as optimum ranks for which the variance/generalized variance is minimum and denote them as $n_1^0, n_2^0, n_3^0, \ldots, n_{(k_1)}^0, n_{(k_1+1)}^0, \ldots, n_{(k_1+k_2)}^0$. Corresponding to these optimum ranks the BLUE's are obtained by computing the 'a' and 'b' coefficients by direct matrix multiplication from the results in [2.2], [2.4], and [2.6]. These computations have been carried out and the optimum ranks and corresponding BLUE's have been obtained for $n=3(1)10$, $k=2(1)5$, and for all possible truncation points R_1 and R_2.

The results have been tabulated for the aforesaid three cases separately, showing the optimum ranks as well as the corresponding coefficients of the BLUE, in a readily usable form. A part of this tabulation is presented in Table 1 and an example follows showing how it can be used for obtaining the estimates of θ_1 and θ_2 as well as the efficiency of such estimates (full tables for $n=3(1)10$, and $k=2(1)5$, are available from the authors).

3. AN EXAMPLE

Consider a censored sample $x_{(1)}, x_{(2)}, x_{(3)}, x_{(6)}$, where $x_{(4)}$ and $x_{(5)}$ are missing. Further, suppose we want to use three of the available observations for estimating unknown θ_1 and θ_2. From Table 1, for $n=6$, $R_1=3$, $R_2=6$, $k=3$, we immediately read out the optimum ranks as $n_1^\circ=1$, $n_2^\circ=3$, $n_3^\circ=6$. The corresponding BLUE's are obtained from the table as—

$$\hat{\theta}_1 = a_{(1)}x_{(1)} + a_{(3)}x_{(3)} + a_{(6)}x_{(6)} = 0.0021\ x_{(1)} + 0.8856\ x_{(3)} + 0.1124\ x_{(6)}$$

$$\hat{\theta}_2 = b_{(1)}x_{(1)} + b_{(3)}x_{(3)} + b_{(6)}x_{(6)} = -\ 0.2644x_{(1)} - 0.0362x_{(3)} + 0.3006x_{(6)}.$$

Also $\dfrac{1}{\theta_2^4}\, GV(\hat{\theta}_1\hat{\theta}_2) = (0.0633)$. If we used all four observations we would have (from Table 1, $k=4$), $\dfrac{1}{\theta_2^4}\, GV(\hat{\theta}_1\hat{\theta}_2) = (0.0599)$. Hence, efficiency of our estimates compared to full available sample is 0.94, which is quite high.

ACKNOWLEDGEMENT

The authors record their deep gratitude to Prof. A. K. Md. E. Saleh Dept. of Mathematics, Carleton University, Ottawa, who suggested the problem and rendered assistance during the work.

The authors are also thankful to the referee for helpful comments.

TABLE 1. *Showing Coefficients of the BLUE's of the Parameters of Double Exponential Distribution with* k*-Optimum Order Statistics from Samples Censored in the Middle (shown here for* n = 4, 5, 6, 7, *and* k = 3,4 *only)*

For $k = 3$

n	R_1	R_2	n_1	n_2	n_3	$a_{(n_1^0)}$ / $b_{(n_1^0)}$	$a_{(n_2^0)}$ / $b_{(n_2^0)}$	$a_{(n_3^0)}$ / $b_{(n_3^0)}$	$\frac{1}{\theta_2^2}V(\hat{\theta}_2)$	$\frac{1}{\theta_2^2}V(\hat{\theta}_1)$	$\frac{1}{\theta_2^2}\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2)$	$\frac{1}{\theta_2^4}GV(\hat{\theta}_1, \hat{\theta}_2)$
4	1	3	1	3	4	0.2103	0.7707	0.0191	0.3127	0.4782	−0.0297	0.1487
						−0.3849	0.0639	0.3210				
4	2	4	1	2	4	0.0191	0.7707	0.2103	0.3127	0.4782	0.0297	0.1487
						−0.3210	−0.0639	0.3849				
5	1	3	1	3	5	0.0670	0.8660	0.0670	0.2547	0.3395	0.0000	0.0865
						−0.3148	0.0000	0.3148				
5	1	4	1	4	5	0.2529	0.7729	−0.0258	0.2496	0.4515	−0.0439	0.1108
						−0.3436	0.0903	0.2533				
5	2	4	1	2	4	0.0119	0.4776	0.5105	0.3009	0.3583	0.0053	0.1078
						−0.3063	−0.2950	0.6013				
5	2	5	1	2	5	−0.0258	0.7729	0.2529	0.2496	0.4515	0.0439	0.1108
						−0.2533	−0.0903	0.3436				
5	3	5	1	3	5	0.0670	0.8660	0.0670	0.2547	0.3395	0.0000	0.0865
						−0.3148	0.0000	0.3148				
6	1	3	1	3	6	0.0021	0.8856	0.1124	0.2162	0.2948	0.0217	0.0633
						−0.2644	−0.0362	0.3006				
6	1	4	1	4	6	0.1124	0.8856	0.0021	0.2162	0.2948	−0.0217	0.0633
						−0.3006	0.0362	0.2644				
6	1	5	1	5	6	0.2820	0.7612	−0.0431	0.2104	0.4481	−0.0504	0.0917
						−0.3139	0.1019	0.2120				
6	2	4	1	4	6	0.1124	0.8856	0.0021	0.2162	0.2948	−0.0217	0.0633
						−0.3006	0.0362	0.2644				
6	2	5	2	5	6	0.5035	0.4912	0.0053	0.2320	0.3369	−0.0016	0.0782
						−0.5096	0.2754	0.2342				
6	2	6	1	2	6	−0.0431	0.7612	0.2820	0.2104	0.4481	0.0504	0.0917
						−0.2120	−0.1019	0.3139				
6	3	5	1	3	6	0.0021	0.8856	0.1124	0.2162	0.2948	0.0217	0.0633
						−0.2644	−0.0362	0.3006				
6	3	6	1	3	6	0.0021	0.8856	0.1124	0.2162	0.2948	0.0217	0.0633
						−0.2644	−0.0362	0.3006				
6	4	6	1	3	6	0.0021	0.8856	0.1124	0.2162	0.2948	0.0217	0.0633
						−0.2644	−0.0362	0.3006				
7	1	3	1	4	7	0.0383	0.9233	0.0383	0.1911	0.2316	0.0000	0.0443
						−0.2632	0.0000	0.2632				
7	1	4	1	4	7	0.0383	0.9233	0.0383	0.1911	0.2316	0.0000	0.0443
						−0.2632	0.0000	0.2632				
7	1	5	1	5	7	0.1483	0.8810	−0.0293	0.1889	0.2837	−0.0348	0.0524
						−0.2859	0.0568	0.2291				
7	1	6	1	6	7	0.3027	0.7477	−0.0504	0.1835	0.4512	−0.0534	0.0799
						−0.2916	0.1074	0.1842				
7	2	4	1	4	7	0.0383	0.9233	0.0383	0.1911	0.2316	0.0000	0.0443
						−0.2632	0.0000	0.2632				
7	2	5	2	5	7	0.3006	0.6981	0.0013	0.1980	0.2520	−0.0092	0.0498
						−0.4962	0.2556	0.2406				
7	2	6	1	2	6	0.0023	0.4963	0.5013	0.1911	0.3286	0.0006	0.0628
						−0.1920	−0.2581	0.4501				

TABLE 1. *Showing Coefficients of the BLUE's of the Parameters of Double Exponential Distribution with* k*-Optimum Order Statistics from Samples Censored in the Middle (shown here for* n = 4,5,6,7, *and* k = 3,4 *only)* — Continued

For $k=3$

n	R_1	R_2	n_1	n_2	n_3	$a_{(n_1^\circ)}$ / $b_{(n_1^\circ)}$	$a_{(n_2^\circ)}$ / $b_{(n_2^\circ)}$	$a_{(n_3^\circ)}$ / $b_{(n_3^\circ)}$	$\frac{1}{\theta_2^2}V(\hat{\theta}_2)$	$\frac{1}{\theta_2^2}V(\hat{\theta}_1)$	$\frac{1}{\theta_2^2}\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2)$	$\frac{1}{\theta_2^4}GV(\hat{\theta}_1, \hat{\theta}_2)$
7	2	7	1	2	7	−0.0504	0.7477	0.3027	0.1835	0.4512	0.0534	0.0799
						−0.1842	−0.1074	0.2916				
7	3	5	2	5	7	0.3006	0.6981	0.0013	0.1980	0.2520	−0.0092	0.0498
						−0.4962	0.2556	0.2406				
7	3	6	1	3	6	0.0013	0.6981	0.3006	0.1980	0.2520	0.0092	0.0498
						−0.2406	−0.2556	0.4962				
7	3	7	1	3	7	−0.0293	0.8810	0.1483	0.1889	0.2837	0.0348	0.0524
						−0.2291	−0.0568	0.2859				
7	4	6	1	4	7	0.0383	0.9233	0.0383	0.1911	0.2316	0.0000	0.0443
						−0.2632	0.0000	0.2632				
7	4	7	1	4	7	0.0383	0.9233	0.0383	0.1911	0.2316	0.0000	0.0443
						−0.2632	0.0000	0.2632				
7	5	7	1	4	7	0.0383	0.9233	0.0383	0.1911	0.2316	0.0000	0.0443
						−0.2632	0.0000	0.2632				

For $k=4$

n	R_1	R_2	n_1	n_2	n_3	n_4	$a_{(n_1^\circ)}$ / $b_{(n_1^\circ)}$	$a_{(n_2^\circ)}$ / $b_{(n_2^\circ)}$	$a_{(n_3^\circ)}$ / $b_{(n_3^\circ)}$	$a_{(n_4^\circ)}$ / $b_{(n_4^\circ)}$	$\frac{1}{\theta_2^2}V(\hat{\theta}_2)$	$\frac{1}{\theta_2^2}V(\hat{\theta}_1)$	$\frac{1}{\theta_2^2}\mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2)$	$\frac{1}{\theta_2^2}GV(\hat{\theta}_1, \hat{\theta}_2)$
5	1	3	1	3	4	5	0.0811	0.7021	0.2123	0.0046	0.2413	0.3286	−0.0121	0.0792
							−0.2991	−0.1821	0.2357	0.2455				
5	2	4	1	2	4	5	0.0173	0.4827	0.4827	0.0173	0.2290	0.3579	0.0000	0.0819
							−0.2331	−0.2264	0.2264	0.2331				
5	3	5	1	2	3	5	0.0046	0.2123	0.7021	0.0811	0.2413	0.3286	0.0121	0.0792
							−0.2455	−0.2357	0.1821	0.2991				
6	1	3	1	3	4	6	0.0273	0.4727	0.4727	0.0273	0.2031	0.2590	0.0000	0.0526
							−0.2491	−0.2861	0.2861	0.2491				
6	1	4	1	4	5	6	0.1173	0.8223	0.0794	−0.0190	0.2063	0.2934	−0.0254	0.0599
							−0.2873	−0.1331	0.2123	0.2081				
6	2	4	1	2	4	6	0.0069	0.2548	0.7111	0.0272	0.1956	0.2732	−0.0005	0.0534
							−0.1974	−0.2493	0.2070	0.2398				
6	2	5	1	2	5	6	0.0069	0.4931	0.4931	0.0069	0.1875	0.3368	0.0000	0.0632
							−0.1893	−0.2234	0.2234	0.1893				
6	3	5	1	3	5	6	0.0272	0.7111	0.2548	0.0069	0.1956	0.2732	0.0005	0.0534
							−0.2398	−0.2090	0.2493	0.1974				
6	3	6	1	2	3	6	−0.0190	0.0794	0.8223	0.1173	0.2063	0.2934	0.0254	0.0599
							−0.2081	−0.2123	0.1331	0.2873				
6	4	6	1	3	4	6	0.0273	0.4727	0.4727	0.0273	0.2031	0.2590	0.0000	0.0526
							−0.2491	−0.2861	0.2861	0.2491				
7	1	3	1	3	5	7	0.0120	0.4880	0.4880	0.0120	0.1681	0.2254	0.0000	0.0379
							−0.2044	−0.2917	0.2917	0.2044				
7	1	4	1	4	6	7	0.0497	0.8385	0.1177	−0.0059	0.1744	0.2273	−0.0084	0.0396
							−0.2407	−0.1686	0.2341	0.1752				
7	1	5	1	5	6	7	0.1491	0.8698	0.0139	−0.0327	0.1815	0.2836	−0.0353	0.0502
							−0.2750	−0.0939	0.1866	0.1823				
7	2	4	1	2	4	7	−0.0059	0.1177	0.8385	0.0497	0.1744	0.2273	0.0084	0.0396
							−0.1752	−0.2341	0.1686	0.2407				

TABLE 1. *Showing Coefficients of the BLUE's of the Parameters of Double Exponential Distribution with* k*-Optimum Order Statistics from Samples Censored in the Middle (shown here for* n = 4,5,6,7, *and* k = 3,4 *only*) — Continued

For $k = 4$

n	R_1	R_2	n_1	n_2	n_3	n_4	$a_{(n\,0)_1}$ / $b_{(n\,0)_1}$	$a_{(n\,0)_2}$ / $b_{(n\,0)_2}$	$a_{(n\,0)_3}$ / $b_{(n\,0)_3}$	$a_{(n\,0)_4}$ / $b_{(n\,0)_4}$	$\frac{1}{\theta_2^2} V(\hat{\theta}_2)$	$\frac{1}{\theta_2^2} V(\hat{\theta}_1)$	$\frac{1}{\theta_2^2} \mathrm{Cov}(\hat{\theta}_1, \hat{\theta}_2)$	$\frac{1}{\theta_2^2} GV(\hat{\theta}_1, \hat{\theta}_2)$
7	2	5	1	2	5	7	0.0099	0.2858	0.7006	0.0037	0.1650	0.2519	−0.0072	0.0415
							−0.1658	−0.2481	0.2134	0.2005				
7	2	6	1	2	6	7	0.0029	0.4971	0.4971	0.0029	0.1602	0.3286	0.0000	0.0526
							−0.1610	−0.2166	0.2166	0.1610				
7	3	5	1	3	5	7	0.0120	0.4880	0.4880	0.0120	0.1681	0.2254	0.0000	0.0379
							−0.2044	−0.2917	0.2917	0.2044				
7	3	6	1	3	6	7	0.0037	0.7006	0.2858	0.0099	0.1650	0.2519	0.0072	0.0415
							−0.2005	−0.2134	0.2481	0.1658				
7	3	7	1	2	3	7	−0.0327	0.0139	0.8698	0.1491	0.1815	0.2836	0.0353	0.0502
							−0.1823	−0.1866	0.0939	0.2750				
7	4	6	1	2	4	7	−0.0059	0.1177	0.8385	0.0497	0.1744	0.2273	0.0084	0.0396
							−0.1752	−0.2341	0.1686	0.2407				
7	4	7	1	2	4	7	−0.0059	0.1177	0.8385	0.0497	0.1744	0.2273	0.0084	0.0396
							−0.1752	−0.2341	0.1686	0.2407				
7	5	7	1	3	5	7	0.0120	0.4880	0.4880	0.0120	0.1681	0.2254	0.0000	0.0379
							−0.2044	−0.2917	0.2917	0.2044				

REFERENCES

[1] Chan, L. K. and N. N. Chan, "Estimation of the Parameters of the Double Exponential Distribution Based on Selected Order Statistics," Bulletin of the Institute of Statistical Research and Training, Dacca University, Vol. 3, No. 2, pp. 21–40 (1969).

[2] Govindarajulu, Z., "Best Linear Estimates Under Symmetric Censoring of the Parameters of a Double Exponential Population," J. Am. Stat. Assoc., Vol. 61, pp. 248–258 (1966).

[3] Lloyd, E. H., "Least Squares Estimation of Location and Scale Parameters Using Order Statistics," Biometrika, Vol. 39, pp. 88–95 (1952).

[4] Sarhan, A. E., "Estimation of the Mean and Standard Deviation by Order Statistics," Annals of Mathematical Statistics, Vol. 25, pp. 317 (1954).

[5] Sarhan, A. E. and B. Greenburg, "Linear Estimates for Doubly Censored Samples from the Exponential Distribution with Observations also Missing from the Middle," Bulletin of the International Statistical Institute, 36th session, Vol. 42, Book 2, pp. 1195–1204 (1967).

THE SINGLE SERVER QUEUE IN DISCRETE TIME-NUMERICAL ANALYSIS IV

David Heimann* and Marcel F. Neuts†

Purdue University

ABSTRACT

The nonlinear difference equation for the distribution of the busy period for an unbounded discrete time queue of $M \mid G \mid 1$ type is solved numerically by a monotone iterative procedure. A starting solution is found by computing a first passage time distribution in a truncated version of the queue.

1. INTRODUCTION

We consider the single server queue in discrete time, studied by Dafermos and Neuts [1]. The numbers of customers forming the queue during successive units of time form a sequence of independent, identically distributed random variables with common probability density $\{p_\nu\}$, where $0 \leq \nu \leq K$. In numerical investigations, we assume that K is a finite positive integer. The service times of the successive customers are independent, identically distributed random variables with common density $\{r_\nu\}$, where $1 \leq \nu \leq L_2$. In numerical investigations, we assume that L_2 is finite. The numbers of arrivals during successive units of time and the successive service times are assumed to be independent sequences of random variables.

In this paper we shall consider the case of *an unbounded* queue, in which there is no upper bound to the number of customers which may be present at any time, and also the case of *a bounded queue.* In the latter case, there is an upper bound L_1 to the number of customers present at any one time. All customers arriving when the queue length is L_1 are lost. For notational simplicity, we write $L_1 = +\infty$, for the unbounded queue.

It is known [1, 3] that both cases may be studied in terms of a bivariate Markov chain $\{(X_n, Y_n), n \geq 0\}$, where X_n denotes the number of customers present at time $n+$ and Y_n denotes the residual service time of the customer in service at time $n+$. We note that $X_n = 0$, if and only if $Y_n = 0$. The Markov chain $\{(X_n, Y_n), n \geq 0\}$ has as its state space, the point $(0, 0)$ and all pairs (i, j), where $1 \leq i \leq L_1, 1 \leq j \leq L_2$.

For purposes of easy reference, we recall that there are four possible types of transitions during a unit of time in the Markov chain $\{(X_n, Y_n), n \geq 0\}$:

(1) When $X_n \geq 1$, $Y_n > 1$ the residual service time $Y_{n+1} = Y_n - 1$, and only arrivals to the queue can occur. $X_{n+1} = \min\ (L_1, X_n + \nu)$, where ν denotes the number of arrivals in the time interval $(n, n+1)$.

---

* The research of this author was sponsored by the Purdue Research Foundation through an XR-Grant.

† Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No. AFOSR-72-2331. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

(2) When $X_n > 1$, $Y_n = 1$, a departure occurs at time $n+1$ and the next customer in line begins service. The residual service time Y_{n+1} is equal to the service time of the customer whose service begins at time $n+1$.

(3) When $X_n = Y_n = 1$, a departure occurs at time $n+1$. At time $n+1$, the queue may become empty if there are no arrivals during the time interval $(n, n+1)$. If there are arrivals during $(n, n+1)$, a new service is initiated at time $n+1$.

(4) When $X_n = Y_n = 0$, the queue is empty at time n. A new service and also a new busy period is started at time $n+1$, if and only if there are arrivals during $(n, n+1)$.

2. THE BUSY PERIOD FOR THE UNBOUNDED QUEUE

We consider a queue with *one* customer at time $n = 0$, whose service is just beginning. We denote by β_n the probability that the queue becomes empty for the first time at time n. The sequence $\{\beta_n\}$ is the probability density of the duration of the busy period. Since by assumption each customer requires at least one unit of time for service, $\beta_0 = 0$.

Denoting by $\{p_\nu^{(k)}\}$ and $\{r_\nu^{(k)}\}$, respectively, the k-fold convolutions of the probability densities $\{p_\nu\}$ and $\{r_\nu\}$ and setting $p_\nu^{(0)} = r_\nu^{(0)} = \delta_{0\nu}$, the sequence $\{\beta_n\}$ satisfies the nonlinear difference equation

$$\beta_n = \sum_{j=1}^{L_2} r_j \sum_{\nu=0}^{\infty} p_\nu^{(j)} \beta_{n-j}^{(\nu)}, \tag{1}$$

for $n \geq 1$. Equation (1) is obtained by application of the law of total probability. The probability that the first service lasts for j units of time and that ν customers arrive during it is $r_j p_\nu^{(j)}$. If $\nu \neq 0$, these ν customers may be considered as the initial customers of ν independent busy periods. The probability that these ν busy periods together take up exactly the $n-j$ remaining units of time is given by $\beta_{n-j}^{(\nu)}$. Equation (1) is recursive, since for each n, the expression on the right involves only $\beta_1, \ldots, \beta_{n-1}$ and the terms of the convolutions of the sequence $\{\beta_\nu\}$ with an index not exceeding $n-1$.

Nevertheless, direct numerical computation of the recurrence relation (1) requires a prohibitive amount of computer memory which seriously limits its practical usefulness.

We note that the second summation in (1) involves only a finite number of terms, since $p_\nu^{(j)}$ is zero for $\nu > Kj$. Even for $j = L_2$, we need to consider only ν such that $0 \leq \nu \leq L_2 K$.

Defining

$$a_0(j) = r_j p_0^j, \quad \text{for } 1 \leq j \leq L_2, \tag{2}$$

$$= 0, \quad \text{elsewhere}$$

and

$$a_\nu(j) = r_j p_\nu^{(j)}, \quad \text{for } 1 \leq j \leq L_2,\ 1 \leq \nu \leq Kj,$$

$$= 0, \quad \text{elsewhere,}$$

we may write Equation (1) in the general form

$$\{\beta_n\} = \sum_{\nu=0}^{KL_2} \{a_\nu\} * \{\beta_n^{(\nu)}\}. \tag{3}$$

The nonlinear difference Equation (1) is therefore of the form

$$\beta = \tilde{P}(\beta), \tag{4}$$

where $\tilde{P}(\beta)$ is a polynomial $a_0 + a_1 * \beta + a_2 * \beta^{(2)} + \ldots + a_{KL_2} * \beta^{(KL_2)}$ in the convolution algebra over the sequences with nonnegative indices. We shall show below that the Equation (4) may be conveniently solved by successive substitutions. We shall then concentrate on the problem of selecting a good starting sequence for this iterative solution.

Before doing so, we shall derive an explicit expression for the probabilities β_n, which is analogous to the explicit formula for the distribution of the busy period in the $M \mid G \mid 1$ queue, due to L. Takács [4].

Denoting the probability generating functions of $\{\beta_n\}$, $\{p_n\}$, and $\{r_n\}$ by $B(z)$, $P(z)$, and $R(z)$, respectively, Equation (1) may be written as:

$$B(z) = R[zP[B(z)]], \qquad |z| \leq 1. \tag{5}$$

The following theorem is well-known and may be proved by a standard method involving Rouché's theorem.

THEOREM 1: For every $|z| < 1$, the functional equation

$$\zeta = R[zP(\zeta)], \tag{6}$$

has a unique root in the unit disk $|\zeta| < 1$. This root $\zeta = B(z)$ is an analytic function of z, which is continuous on the boundary $|z| = 1$. The quantity $\theta = B(1-)$ is the smallest positive root of the equation

$$\zeta = R[P(\zeta)]. \tag{7}$$

Moreover $\theta = 1$, if and only if $P'(1-)R'(1-) \leq 1$, and $0 < \theta < 1$, if and only if $P'(1-)R'(1-) > 1$.

If $\zeta = P'(1-)R'(1-) < 1$, we say that the queue is *stable*. The irreducible, aperiodic Markov chain $\{(X_n, Y_n), n \geq 0\}$ is then *positive recurrent*. If $\zeta = 1$, we say that the queue is *critical*; the Markov chain is then *null-recurrent*. If $\zeta > 1$, the queue is *unstable*. The probability that the busy period does not end is $1 - \theta > 0$.

For a stable queue, the busy period has moments of all orders. Klimko and Neuts [2] have shown that moments up to order 50 may be computed numerically by repeated applications of Fáa di Bruno's formula.

From Equation (6), an explicit formula for $B(z)$ may be derived by use of Lagrange's Inversion Formula [6]. We first modify the functional Equation (5) by means of the inverse function $R^{-1}(\cdot)$. For every ζ, $|\zeta| < 1$, $R(\zeta)$ is defined as that branch of the solution of $R[R^{-1}(\zeta)] = \zeta$, which is *real* for

$0 \leq \zeta < 1$. $R^{-1}(\cdot)$ is then analytic in $|\zeta| < 1$. Setting $R^{-1}[B(z)] = w(z)$, Equation (5) may be rewritten as:

$$w(z) = zP[R(w(z))], \tag{8}$$

for $|z| \leq \delta \leq 1$. Again by Rouché's theorem, we obtain that the functional equation

$$\zeta = zP[R(\zeta)], \qquad |z| \leq \delta < 1, \tag{9}$$

has a unique solution inside $|\zeta| \leq 1$, for every z, with $|z| \leq \delta$. The function $B(z)$ is then given by

$$B(z) = R(\zeta) = R[w(z)]. \tag{10}$$

Applying Lagrange's Inversion Formula to (9), it follows that for all z with $|z| \leq \delta$, we have

$$B(z) = R(0) + \sum_{n=1}^{\infty} \frac{z^n}{n!} \left\{ \frac{d^{n-1}}{ds^{n-1}} \{R'(s)[P[R(s)]]^n\} \right\}_{s=0}. \tag{11}$$

We recall that $R(0) = 0$, so that $\beta_0 = 0$. For $n \geq 1$, we have

$$\begin{aligned} \beta_n &= \frac{1}{n!} \left\{ \frac{d^{n-1}}{ds^{n-1}} \{R'(s)[P[R(s)]]^n\} \right\}_{s=0} \qquad (12) \\ &= \frac{1}{n!} \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \left[\frac{d^{\nu+1} R(s)}{ds^{\nu+1}} \right]_{s=0} \left[\frac{d^{n-1-\nu}}{ds^{n-1-\nu}} P^n[R(s)] \right]_{s=0}, \end{aligned}$$

by Leibniz's Rule. We note that

$$\left[\frac{d^{\nu+1} R(s)}{ds^{\nu+1}} \right]_{s=0} = (\nu+1)! r_{\nu+1}, \qquad \text{for } 0 \leq \nu \leq L_2 - 1, \tag{13}$$

and zero for $\nu \geq L_2$, so that Formula (12) may be rewritten as:

$$\beta_n = \frac{1}{n} \sum_{\nu=0}^{n-1} (n-\nu) r_{n-\nu} \frac{1}{\nu!} \left[\frac{d^\nu}{ds^\nu} P^n[R(s)] \right]_{s=0}. \tag{14}$$

The derivatives of $P^n[R(s)]$ are given by Faá di Bruno's Formula as follows:

$$\left[\frac{d^\nu}{ds^\nu} P^n[R(s)] \right]_{s=0} = \sum_{t=1}^{\nu} \frac{n!}{(n-t)!} \sum_{\substack{j_1+\ldots+j_\nu=t \\ j_1+\ldots+\nu j_\nu=\nu}} \frac{\nu!}{j_1! \ldots j_\nu!} \theta_1^{j_1} \ldots \theta_\nu^{j_\nu}, \tag{15}$$

for $1 \leq \nu \leq n$, where,

$$\theta_i = \frac{1}{i!}\left[\frac{d^i P[R(s)]}{ds^i}\right]_{s=0} = \sum_{k=0}^{K} p_k r_i^{(k)}, \tag{16}$$

for $i \geq 1$, since θ_i is the coefficient of s^i in the power series expansion of $P[R(s)]$. Combining Formulas (14), (15), and (16) we obtain the following explicit expression for β_n, $n \geq 1$:

$$\beta_n = r_n p_0^n + (n-1)! \sum_{\nu=1}^{n-1} (n-\nu) r_{n-\nu} \sum_{t=1}^{\nu} \frac{1}{(n-t)!} \sum_{\substack{j_1+\ldots+j_\nu = t \\ j_1+2j_2+\ldots+\nu j_\nu = \nu,}} \frac{\theta_1^{j_1}}{j_1!} \cdots \frac{\theta_\nu^{j_\nu}}{j_\nu!}, \tag{17}$$

where θ_i is given by (16).

Although for every n, the right hand side in Formula (17) is a finite sum, it is not a suitable expression for numerical computation. The number of terms in it grows very rapidly for higher values of n. Using the efficient algorithm developed by E. Klimko [2] for generating the indices j_i, $i = 1, \ldots, \nu$ in the last sum, one can compute at most the first 50 terms of the sequence $\{\beta_n\}$ without running into prohibitive computation times and memory storage requirements.

3. THE ITERATIVE PROCEDURE

The fixed point Equation (4) suggests that the nonlinear difference equation for the sequence $\{\beta_n\}$ may lend itself well to a solution by successive substitutions. We shall prove that this is the case and furthermore that it is possible to find a sequence of solutions which converges *monotonically* to the density $\{\beta_n\}$, in a sense to be defined below.

We shall consider the set $\mathscr{B}$ of all sequences $\underline{b} = \{b_n, n \geq 0\}$ for which $b_n \geq 0$ for all $n \geq 0$, and $\sum_{n=0}^{\infty} b_n \leq 1$. The convolution product $b * c$ of any two sequences in $\mathscr{B}$ is itself in $\mathscr{B}$. We denote by $b^{(k)}$ the k-fold convolution of $\underline{b}$ with itself and we define $\underline{b}^{(0)}$ to be the sequence $(1, 0, 0, \ldots)$.

To every sequence in $\mathscr{B}$ we associate the sequence $\underline{B} = \{B_n, n \geq 0\}$, where $B_n = \sum_{\nu=0}^{n} b_\nu$, for $n \geq 0$.

We further define the partial ordering $\overset{\mathscr{L}}{\prec}$ on the set $\mathscr{B}$, as follows:

$$\underline{b} \overset{\mathscr{L}}{\prec} \underline{c} \text{ if and only if } B_n \leq C_n, \qquad \text{for all } n \geq 0. \tag{18}$$

It is straightforward to verify that if $\underline{a}, \underline{b}, \underline{c} \in \mathscr{B}$, then

$$\underline{a} \overset{\mathscr{L}}{\prec} \underline{b} \text{ implies } \underline{a} * \underline{c} \overset{\mathscr{L}}{\prec} \underline{b} * \underline{c}. \tag{19}$$

LEMMA 1: If $\underline{a}, \underline{b} \in \mathscr{B}$, then $\underline{a} \overset{\mathscr{L}}{\prec} \underline{b}$ implies $\underline{a}^{(k)} \overset{\mathscr{L}}{\prec} \underline{b}^{(k)}$, for $k \geq 0$.

PROOF: The statement is clearly true for $k \leq 1$. Assuming it to hold for any k, we have

$$a^{(k+1)} = \underline{a}^{(k)} * \underline{a} \overset{\mathscr{L}}{\prec} \underline{b}^{(k)} * \underline{a} = \underline{a} * \underline{b}^{(k)} \overset{\mathscr{L}}{\prec} \underline{b} * \underline{b}^{(k)} = \underline{b}^{(k+1)}.$$

The lemma holds therefore by induction on k.

We now consider a sequence of densities ${}_\nu\underline{a}$, $\nu \geq 0$, belonging to $\mathscr{B}$ such that

$$\sum_{\nu=0}^{\infty} {}_\nu\underline{a} \in \mathscr{B},$$

and we form the convolution series

$$A(\underline{b}) = \sum_{\nu=0}^{\infty} {}_\nu\underline{a} * \underline{b}^{(\nu)}. \tag{20}$$

THEOREM 2: For $\underline{b} \in \mathscr{B}$, ${}_\nu\underline{a} \in \mathscr{B}$, for $\nu \geq 0$ and

$$\sum_{\nu=0}^{\infty} {}_\nu a \in \mathscr{B},$$

we have that $A(\underline{b}) \in \mathscr{B}$. Furthermore $\underline{b} \overset{\mathfrak{L}}{\prec} \underline{c}$ implies that $A(\underline{b}) \overset{\mathfrak{L}}{\prec} A(\underline{c})$.

PROOF: The terms of the sequence $A(\underline{b})$ are clearly nonnegative. Moreover, denoting

$$T(\underline{b}) = \sum_{n=0}^{\infty} \underline{b}_n,$$

we have that $T(\underline{b} * \underline{c}) = T(\underline{b})T(\underline{c})$.

It follows that

$$T[A(\underline{b})] = \sum_{\nu=0}^{\infty} T({}_\nu\underline{a})T^\nu(\underline{b}) \leq \sum_{\nu=0}^{\infty} T({}_\nu\underline{a}) \leq 1.$$

Furthermore $\underline{b} \overset{\mathfrak{L}}{\prec} \underline{c}$ implies that ${}_\nu\underline{a} * \underline{b}^{(\nu)} \overset{\mathfrak{L}}{\prec} {}_\nu\underline{a} * \underline{c}^{(\nu)}$, for all $\nu \geq 0$ and therefore by summation on ν, $A(\underline{b}) \overset{\mathfrak{L}}{\prec} A(\underline{c})$.

We define the iterative process by the following relation, derived from Equation (3):

$${}_k\underline{\beta} = \sum_{\nu=0}^{L \geq K+} \underline{a}_\nu * {}_{k-1}\underline{\beta}^{(\nu)}, \qquad \text{for } k \geq 1, \tag{21}$$

where the sequences $\underline{a}_\nu$ are defined in Equation (2). The sequences ${}_k\underline{\beta}$ are the successive iterates and the sequence ${}_0\underline{\beta}$ is the starting solution.

Since the right hand side of Equation (21) is of the form $A({}_{k-1}\underline{\beta})$, we use Theorem 2 to prove the convergence of the iterative process.

THEOREM 3: Let $\{{}_k\underline{b}\}$, $k \geq 0$, be a sequence of densities such that ${}_0\underline{b} \in \mathscr{B}$, ${}_1\underline{b} \overset{\mathfrak{L}}{\prec} {}_0\underline{b}$, and ${}_{k+1}\underline{b} = A({}_k\underline{b})$ for $k \geq 0$. Then $\{{}_k\underline{b}\}$ converges monotonically to a unique limit $\underline{\beta} \in \mathscr{B}$.

PROOF: Clearly ${}_k\underline{b} \in \mathscr{B}$ for $k \geq 0$, by Theorem 2. Let ${}_k\underline{b}' = {}_{k+1}\underline{b}$. Then ${}_0\underline{b}' \overset{\mathfrak{L}}{\prec} {}_0\underline{b}$, and if moreover ${}_k\underline{b}' \overset{\mathfrak{L}}{\prec} {}_k\underline{b}$, then by Theorem 2 $A({}_k\underline{b}') \overset{\mathfrak{L}}{\prec} A({}_k\underline{b})$, or ${}_{k+1}\underline{b}' \overset{\mathfrak{L}}{\prec} {}_{k+1}\underline{b}$. Therefore, by induction on k, we

have ${}_{k+1}\underline{b} \overset{\mathfrak{L}}{\prec} {}_k\underline{b}$, for all $k \geq 0$. Convergence follows from the fact that the sequence (0, 0, . . .) is a uniform lower bound for all of the ${}_k\underline{b}$'s. Because of the monotone convergence, the limit $\underline{\beta}$ is also in $\mathscr{B}$. This limit $\underline{\beta}$ is unique, since clearly $\underline{\beta} = A(\underline{\beta})$ by passage to the limit in (21) and since Equation (17) admits of a unique solution.

A similar monotonicity result can be obtained when ${}_0\underline{b} \overset{\mathfrak{L}}{\prec} {}_1\underline{b}$. Even without the restriction ${}_1\underline{b} \overset{\mathfrak{L}}{\prec} {}_0\underline{b}$, we have:

THEOREM 4: If $\{{}_k\underline{\beta}\}$, $k \geq 0$, is a sequence of densities, such that ${}_{k+1}\underline{\beta} = A({}_k\underline{\beta})$ and ${}_0\underline{\beta} \in \mathscr{B}$, then $\{{}_k\underline{\beta}\}$ converges termwise to $\underline{\beta}$.

PROOF: Consider the sequences of iterates $\{{}_k\underline{b}\}$ and $\{{}_k\underline{b}^*\}$, where ${}_{k+1}\underline{b} = A({}_k\underline{b})$, ${}_{k+1}\underline{b}^* = A({}_k\underline{b}^*)$ for $k \geq 0$, and ${}_0\underline{b} = (1, 0, 0, \ldots)$, ${}_0\underline{b}^* = (0, 0, 0, \ldots)$. Since ${}_1\underline{b} \overset{\mathfrak{L}}{\prec} {}_0\underline{b}$ and ${}_0\underline{b}^* \overset{\mathfrak{L}}{\prec} {}_1\underline{b}^*$, Theorem 3 implies that both sequences converge monotonically to $\underline{\beta}$. Also, since ${}_0\underline{b}^* \overset{\mathfrak{L}}{\prec} {}_0\underline{\beta} \overset{\mathfrak{L}}{\prec} {}_0\underline{b}$, we have ${}_k\underline{b}^* \overset{\mathfrak{L}}{\prec} {}_k\underline{\beta} \overset{\mathfrak{L}}{\prec} {}_k\underline{b}$ for all $k \geq 0$. Therefore $\{{}_k\underline{\beta}\}$ converges to $\underline{\beta}$.

REMARK: The starting solution ${}_0\underline{b}$ and ${}_0\underline{b}^*$ are readily available and yield monotone sequences of approximants. However, since each iteration involves a large number of elementary operations, it is clearly to our advantage to obtain first a starting solution closer to the sequence $\underline{\beta}$ satisfying (17). Such starting solutions may be obtained by calculating the distribution of the busy period for a *bounded* queue, having the same arrival and service distributions as the unbounded queue under consideration.

4. THE BUSY PERIOD FOR THE BOUNDED QUEUE

The probability density of the busy period for the bounded queue is found by a standard investigation of the absorption time distribution in a finite Markov chain. We consider a queue with upper bound L_1 starting at time $n = 0$, with i customers and a residual service time j, where $1 \leq i \leq L_1$, $1 \leq j \leq L_2$.

$G_n(i, j)$ denotes the probability that the queue with upper bound L_1 becomes empty no later than time n, given the initial state (i, j). The probability $g_n(i, j)$ that the queue ends exactly at time n is then given by the difference $G_n(i, j) - G_{n-1}(i, j)$.

If $\beta_n(L_1)$ denotes the probability that a queue with upper bound L_1 and one customer initially, becomes empty for the first time at time n, and $B_n(L_1)$ denotes the probability that such a queue becomes empty for the first time no later than time n, then

$$\beta_n(L_1) = \sum_{j=1}^{L_2} r_j g_n(1, j), \qquad \text{for } n \geq 1, \tag{22}$$

and

$$B_n(L_1) = \sum_{m=1}^{n} \beta_m(L_1) = \sum_{j=1}^{L_2} r_j G_n(1, j). \tag{23}$$

In order to evaluate the probability density $\{\beta_n(L_1)\}$, it suffices therefore to compute the conditional probabilities $G_n(i, j)$.

The probabilities $G_n(i, j)$ satisfy the following recurrence relations:

$$G_n(i, j) = \sum_{\nu=0}^{K} p_\nu G_{n-1}[\min\,(i+\nu, L_1), j-1], \tag{24}$$

for $i \geq 1, j > 1$.

$$G_n(i, 1) = \sum_{k=1}^{L_2} r_k \sum_{\nu=0}^{K} p_\nu G_{n-1}[\min(i+\nu-1, L_1), k],$$

for $i \geq 2, j = 1$.

$$G_n(1, 1) = p_0 + \sum_{k=1}^{L_2} r_k \sum_{\nu=1}^{K} p_\nu G_{n-1}[\min(\nu, L_1), k]$$

with the initial conditions

$$G_0(i, j) = \delta_{i0}\delta_{j0}. \tag{25}$$

The recurrence relations (24) are immediate by consideration of the possible states of the Markov chain after one unit of time.

In the event relating to the probability $G_n(i, j)$, it is possible that the queue length attains the upper bound L_1 before the queue becomes empty for the first time. In order to study the effect of imposing an upper bound to the length of the queue, we shall also consider the taboo probabilities $\tilde{G}_n(i, j)$, defined as follows: $\tilde{G}_n(i, j)$ is the conditional probability that a queue with initial state (i, j) becomes empty no later than time n, without the queue length exceeding $L_1 - 1$ in between.

The probabilities $\tilde{G}_n(i, j)$ satisfy the recurrence relations

$$\tilde{G}_n(i, j) = \sum_{\nu=0}^{\min(K, L_1-1-i)} p_\nu \tilde{G}_{n-1}(i + \nu, j - 1), \tag{26}$$

for $i \geq 1, j > 1$ and

$$\tilde{G}_n(i, 1) = \sum_{k=1}^{L_2} r_k \sum_{\nu=0}^{\min(K, L_1-i)} p_\nu \tilde{G}_{n-1}(i + \nu - 1, k),$$

for $i \geq 2, j = 1$, and

$$\tilde{G}_n(1, 1) = p_0 + \sum_{k=1}^{L_2} r_k \sum_{\nu=1}^{\min(K, L_1-1)} p_\nu \tilde{G}_{n-1}(\nu, k),$$

for $n \geq 1$, with the initial conditions

$$\tilde{G}_0(i, j) = \delta_{i0}\delta_{j0}. \tag{27}$$

For a given L_1, the quantity

$$\tilde{\beta}_n(L_1) = \sum_{j=1}^{L_2} r_j [\tilde{G}_n(1, j) - \tilde{G}_{n-1}(1, j)], \tag{28}$$

is the probability that a busy period with one customer initially ends no later than time n and that the maximum queue length during that busy period is *less* than L_1. The probability $\tilde{\beta}_n(L_1)$ is of course the

same for the unbounded queue as for a bounded queue whose bound is at least L_1. Similarly to Equation (23), we also define

$$\tilde{B}_n(L_1) = \sum_{m=1}^{n} \tilde{\beta}_m(L_1) = \sum_{j=1}^{L_2} r_j \tilde{G}_n(1,j). \tag{29}$$

THEOREM 5: For all i_1, i_2, j satisfying $0 \leq i_1 \leq i_2 \leq L_1, 0 \leq j \leq L_2$, we have

$$G_n(i_1,j) \geq G_n(i_2,j), \tag{30}$$

for all $n \geq 0$.

PROOF: The inequality clearly holds for $n=0$. Using formula (24) we obtain

(31)

$$G_n(i_1,j) = \sum_{\nu=0}^{K} p_\nu G_{n-1}[\min(i_1+\nu, L_1), j-1] \geq \sum_{\nu=0}^{K} p_\nu G_{n-1}[\min(i_2+\nu, L_1), j-1] = G_n(i_2,j),$$

for $i_1 \geq 1, j > 1$. A similar argument holds in the cases $i_1 > 1$, $i_2 > 1, j=1$, and $i_1=1$, $i_2 \geq 1, j=1$. The arguments for the cases $i_1=j=0$, $i_2>0$ and $i_1=i_2=j=0$ are obvious. The result of Theorem 5 also holds for the probabilities $\tilde{G}_n(i,j)$.

In Theorems 6, 9, and 10 below, we consider two different upper bounds L_1' and L_1''. In order to indicate which upper bound is used, we write $G_{n,L_1'}(i,j)$ and $\tilde{G}_{n,L_1'}(i,j)$ for $G_n(i,j)$ and $\tilde{G}_n(i,j)$, respectively, and correspondingly for L_1''.

THEOREM 6: $L_1' \leq L_1''$ implies that

$$\tilde{G}_{n,L_1'}(i,j) \leq \tilde{G}_{n,L_1''}(i,j), \tag{32a}$$

and

$$G_{n,L_1'}(i,j) \geq G_{n,L_1''}(i,j), \tag{32b}$$

for $0 \leq i \leq L_1'$, $0 \leq j \leq L_2$, and $n \geq 0$.

PROOF: The proof is by induction on n. The statements are obvious when $n=0$. Assume them to hold for $n-1$. We shall prove only the case where $i \geq 1, j > 1$ in detail. The other cases are similar.

The following inequalities prove (32a):

$$\tilde{G}_{n,L_1'}(i,j) = \sum_{\nu=0}^{\min(K,L_1'-i-1)} p_\nu \tilde{G}_{n-1,L_1'}(i+\nu, j-1) \leq \sum_{\nu=0}^{\min(K,L_1'-i-1)} p_\nu \tilde{G}_{n-1,L_1''}(i+\nu, j-1)$$

$$\leq \sum_{\nu=0}^{\min(K,L_1''-i-1)} p_\nu \tilde{G}_{n-1,L_1''}(i+\nu, j-1) = \tilde{G}_{n,L_1''}(i,j).$$

The first inequality follows by the induction hypothesis.

Inequality (32b) is proved by means of the following:

$$G_{n,L_1'}(i,j) = \sum_{\nu=0}^{K} p_\nu G_{n-1,L_1'}(\min\ (i+\nu, L_1'), j-1) \geq \sum_{\nu=0}^{K} p_\nu G_{n-1,L_1''}(\min\ (i+\nu, L_1'), j-1)$$

$$\geq \sum_{\nu=0}^{K} p_\nu G_{n-1,L_1''}(\min\ (i+\nu, L_1''), j-1)$$

$$= G_{n,L_1''}(i,j).$$

The first inequality follows from the induction hypothesis and the second is a consequence of Theorem 5.

Theorem 7 now follows immediately from Theorem 6 and the definitions of $\tilde{\beta}_n(L_1)$ and $\underline{\beta}_n(L_1)$.

THEOREM 7: If $L_1' \leq L_1''$, then for $n \geq 0$,

$$\tilde{B}_n(L_1') \leq \tilde{B}_n(L_1''), \tag{33a}$$

$$B_n(L_1') \geq B_n(L_1''). \tag{33b}$$

The inequalities in Theorem 7 are of course intuitive. As the bound L_1 increases, it becomes more probable that the queue becomes empty before reaching the queue length L_1. It also becomes less probable that the queue empties out before time $n+1$, because more customers may be allowed to join the queue.

5. THE PROBABILITY OF TERMINATION OF THE BUSY PERIOD

If the queue is unstable, i.e., $\rho > 1$, the duration of the busy period is infinite with positive probability. Denoting by θ, the probability that the busy period ends in finite time, we note that θ is given by the smallest positive root of the equation

$$\theta = R[P(\theta)]. \tag{34}$$

The quantity θ is readily computed by successive substitutions or by Newton's method. Whenever $\rho > 1$, the probability θ is first computed. In the alternate case, $\theta = 1$.

In general, the queues for which the traffic intensity ρ is close to one present computational difficulties, because of the long tail of the distribution of the busy period. In such cases, a large number of the probabilities β_n need to be computed to obtain a detailed picture of the busy period. In the sequel, we shall use the term "near-critical" for those cases where $|1-\rho| \leq 0.1$.

6. STOPPING CRITERIA

In any practical use, the recursive or iterative procedure must stop after a finite number of steps. A major question is to determine when it is best to terminate. We shall now consider this problem for both the recursive process (for the bounded queue) and the iterative process (for the unbounded queue).

6.1 The Bounded Queue

We stop at a value N such that

$$\frac{1}{m}\sum_{i=1}^{m}\beta_{N-m+i}(L_1) < \epsilon, \tag{35a}$$

and

$$|\theta - B_N(L_1)| < \delta, \tag{35b}$$

where m, ϵ, and δ are given. Equation (35a) is the condition that an average of m consecutive density values, ranging from $\beta_{N-m+1}(L_1)$ to $\beta_N(L_1)$, be small, while (35b) requires that the cumulative probability be close to its limit θ. Clearly, the value of N will be higher in the case of near-critical queues than for other types of queues.

It should be pointed out that in actuality, the cumulative probability of the truncated unstable queue converges to 1, rather than the θ defined in (34). However, in the cases which we have examined, the cumulative probability tends to stay close to θ for many time points, and thereafter approaches one very slowly. We may therefore use θ as the critical value as discussed in the preceding paragraph. This is plausible because the only way the busy period of the truncated queue can end earlier than that of the unbounded queue is for it to empty out despite its length having reached the upper bound L_1. The probability of this happening within a reasonable time is very small, since the queue is inherently unstable.

6.2 The Unbounded Queue

In this case, the termination criterion is that the difference between two successive iterates for the cumulative distribution function be very small. We therefore stop when:

$$\max_{1\le m\le N} |{}_kB_m - {}_{k-1}B_m| < \epsilon, \tag{36}$$

where ${}_k\underline{B}$ and ${}_{k-1}\underline{B}$ are the successive iterates and ϵ and N are given. The value of k will again be higher for near-critical queues than for other types.

7. APPROXIMATION OF THE UNBOUNDED QUEUE

The busy period probabilities of the bounded queue may be used to approximate those of the unbounded queue. The approximation is quite good, and it saves the rather large amount of computer time required by the iterative process. We proceed to develop the approximation.

Let $G_n^{(u)}(i, j)$ be the probability than an *unbounded* queue becomes empty no later than time n. The recurrence relations for $G_n^{(u)}(i, j)$ may be obtained from Equation (24) by letting $L_1 \to \infty$. Clearly

$$B_n = \sum_{j=1}^{L_2} r_j G_n^{(u)}(1, j), \tag{37}$$

where B_n is the cumulative distribution function of the busy period for the unbounded queue.

The following theorem is similar to Theorem 5. Its proof is therefore omitted.

THEOREM 8: For all i_1, i_2 and j, such that, $0 \leq i_1 \leq i_2 \leq L_1$, and $0 \leq j \leq L_2$, we have

$$G_n^{(u)}(i,j) \geq G_n^{(u)}(i_2, j),$$

for all $n \geq 0$.

The following theorems establish the approximation:

THEOREM 9: For any L_1', L_1'', and $0 \leq i \leq \min(L_1', L_1'')$, $1 \leq j \leq L_2$, and $n \geq 0$, we have $G_{n,L_1'}(i,j) \leq G_n^{(u)}(i,j) \leq G_{n,L_1''}(i,j)$.

PROOF: We consider the case where $i \geq 1$ and $j > 1$. The other cases are similarly proved.

The proof is by induction on n. The theorem is obviously true if $n = 0$. Assume it is true for $n-1$. Then

$$\tilde{G}_{n,L_1'}(i,j) = \sum_{\nu=0}^{\min(K, L_1'-i-1)} p_\nu \tilde{G}_{n-1,L_1'}(i+\nu, j-1) \leq \sum_{\nu=0}^{K} p_\nu \tilde{G}_{n-1,L_1'}(i+\nu, j-1)$$

$$\leq \sum_{\nu=0}^{K} p_\nu G_{n-1}^{(u)}(i+\nu, j-1) = G_n^{(u)}(i,j),$$

where the last inequality follows by the inductive hypothesis.

Furthermore,

$$G_n^{(u)}(i,j) = \sum_{\nu=0}^{K} p_\nu G_{n-1}^{(u)}(i+\nu, j-1) \leq \sum_{\nu=0}^{K} p_\nu G_{n-1}^{(u)}(\min(i+\nu, L_1''), j-1)$$

$$\leq \sum_{\nu=0}^{K} p_\nu G_{n-1,L_1''}(\min(i+\nu, L_1''), j-1) = G_{n,L_1''}(i,j).$$

The first inequality is justified by Theorem 8, while the second follows from the inductive hypothesis.

THEOREM 10: For any L_1', L_1'', and for $n \geq 0$, the following inequality holds:

$$\tilde{B}_n(L_1') \leq B_n \leq B_n(L_1'').$$

PROOF: The theorem follows immediately from Theorem 9 and Equations (23), (29), and (37).

B_n is therefore known to lie within the interval $[\tilde{B}_n(L_1), B_n(L_1)]$, for any given L_1. As L_1 increases, the interval length, which we denote by ϵ_n, becomes smaller, and thus a better approximation.

For queues which are not near-critical, ϵ_n is quite small, usually less than 10^{-5}. This can be explained by the fact that ϵ_n represents the probability that the queue empties out before time n, despite its length reaching the upper bound L_1 at least once before this occurrence. In the case of a stable queue, the probability of ever reaching L_1 is very small. For an unstable queue, while the probability of reaching L_1 is high, that of emptying out subsequently is very small.

However, the approximation is not as good for near-critical queues. This is because both the above probabilities are now significant.

8. COMPUTATIONAL ASPECTS OF THE UNBOUNDED QUEUE

Consider once again Equation (21) for the iterative process

$$_k\underline{\beta} = \sum_{\nu=0}^{L_2K} \underline{a}_\nu * {}_{k-1}\underline{\beta}^{(\nu)}.$$

This is more readily evaluated by use of Horner's method

$$(38) \qquad {}_k\underline{\beta} = \underline{a}_0 + {}_{k-1}\underline{\beta} * (\underline{a}_1 + {}_{k-1}\underline{\beta} * (\underline{a}_2 + \ldots + {}_{k-1}\underline{\beta} * (a_{L_2K-1} + {}_{k-1}\underline{\beta} * \underline{a}_{L_2K}) \ldots)).$$

The advantage of (38) is that it requires only two arrays of length N (one for ${}_k\underline{\beta}$ and one for ${}_{k-1}\underline{\beta}$), whereas (21) requires three (one for ${}_k\underline{\beta}$, one for ${}_{k-1}\underline{\beta}$, and one for the intermediate storage of the arrays ${}_{k-1}\underline{\beta}^{(\nu)}$ as they are calculated and used in the equation).

We now take a count of the number of multiplications required to compute ${}_k\underline{\beta}$. This is a reasonable indicator of the computation time necessary for an algorithm, since multiplications take a great deal more time than additions and so have the largest effect on the total computation time.

The first quantity computed is ${}_{k-1}\underline{\beta} * \underline{a}_{L_2K}$. Since $(\underline{a}_{L_2K})_i = 0$ except for $i = L_2$, the above quantity needs $N - L_2$ multiplications for its calculation. Let $\underline{\gamma}_{L_2K-1} = \underline{a}_{L_2K-1} + {}_{k-1}\underline{\beta} * \underline{a}_{L_2K}$, and $\underline{\gamma}_\nu = \underline{a}_\nu + {}_{k-1}\underline{\beta} * \underline{\gamma}_{\nu+1}$, $\nu = 0, 1, \ldots, L_2K-2$, so that $\underline{\gamma}_0 = {}_k\underline{\beta}$. To compute $\underline{\gamma}_\nu$ requires $N(N-1)/2$ multiplications, since $(\underline{\gamma}_\nu)_i = (\underline{a}_\nu)_i + \sum_{j=1}^{i-1} ({}_{k-1}\underline{\beta})_j (\underline{\gamma}_{\nu+1})_{i-j}$ requires $i-1$ multiplications, for $i = 2, 3, \ldots, N$. Therefore, to compute $\underline{\gamma}_0 = {}_k\underline{\beta}$, a total of

$$(39) \qquad MC = \frac{N(N-1)}{2} (L_2K - 1) + (N - L_2)$$

multiplications are required.

We use the multiplications count (39) to ascertain the sensitivity of the computation time to N, L_2, and K. Since the first term of (39) dominates the second, the value of MC, and thus the computation time, is approximately proportional to the second power of N, and the first power of L_2 and K. The computation is therefore most sensitive to N.

To find an estimate for the total machine time required to run the program for any given value of N, we execute it twice for two relatively low values of N, and use the run times for them to find the values a and b in the following equation derived from (39):

$$(40) \qquad T = aiN(N-1) + b,$$

where T is the estimated run time, i is the number of iterations executed by the program, a is a constant of proportionality, and b is that part of the run time not spent in iteration, e.g., input and output, and computing the initial estimate.

As an example, we consider an unstable queue with $L_2 = 6$ and $K = 2$. The two trial runs were carried out with $N = 100$ and $N = 250$, and the run times were 4 and 11 seconds, respectively. Each run required one iteration. We obtain

$$4 = 9900a + b$$

$$11 = 62250a + b,$$

which yields the solution, $a=0.0001337$, $b=2.676$ sec. Using these values, we calculate the following estimates for the computation time:

N	*Estimated time*	*Actual time*
100	4	4
250	11	11
500	36	37
650	59	63

These results give a good estimate of the maximum value of N which can be handled within a given time limit.

Equation (40) is a special case of a general equation relating the estimated computation time to N, L_1, L_2, K, and M (the number of time points for which the initial estimate is calculated):

$$T=aiN(N-1)L_2K+bML_1L_2K+c. \tag{41}$$

The first term of (41) represents the time used by the iterative process itself; the second term represents the time for computing the initial estimate; and the third represents miscellaneous "housekeeping" time (which is usually quite small).

A technical report, containing program listings, numerical examples, and processing time data is available from the authors upon request.

REFERENCES

[1] Dafermos, S. and M. F. Neuts, "A Single Server Queue in Discrete Time," Cahiers du Centre de Recherche Opérationnelle, 13, 23–40 (1971).

[2] Klimko, E. and M. F. Neuts, "The Single Server Queue in Discrete Time-Numerical Analysis II," in Naval Research Logistics Quarterly, *20*, 297–304 1973.

[3] Neuts, M. F.,"The Single Server Queue in Discrete Time-Numerical Analysis I," Naval Research Logistics Quarterly, *20*, 305–319 1973.

[4] Takács, L., *Introduction to the Theory of Queues* (Oxford University Press, New York, 1962).

[5] Titchmarsh, E. C., *The Theory of Functions* (Oxford University Press, 1939).

[6] Whittaker, E. T. and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, 1927).

ALTERNATE METHODS OF PROJECT SCHEDULING WITH LIMITED RESOURCES

James H. Patterson

College of Business Administration
The Pennsylvania State University

ABSTRACT

The applicability of critical path scheduling is limited by the inability of the algorithm to cope with conflicting resource demands. This paper is an assessment of the effectiveness of many of the heuristic extensions to the critical path method which resolve the conflicts that develop between the resources demanded by an activity and those available. These heuristic rules are evaluated on their ability to solve a large multiproject scheduling problem.

INTRODUCTION

PERT and CPM scheduling techniques implicitly assume that resources are available to schedule activities at their technologically determined start times. Those who are familiar with these techniques are aware, however, that activities often demand more resources than are available. When this happens, a decision must be made regarding which activities to delay and which activities to schedule.

This constrained-resource problem has not yielded readily to the solution techniques of mathematical programming. Attempts to solve the resource-constrained, project scheduling problem by linear [27], integer [2], quadratic [1], and zero-one programming [21], as well as by bounded enumeration [4] and by branch and bound [10] have been successful only on small sets of moderate size projects.* For the majority of practical project scheduling problems, the methods are inappropriate.

Because this problem is not easily solved by the available mathematical programming routines, numerous efforts [15], [16], [27] have been expended in developing elaborate computer-based heuristic solution procedures in order to determine feasible solutions to the problem. To date, the more computationally practical programs have been those employing heuristic methods of solution. In Wiest's latest version of his SPAR–2 program, for example, a problem with 1,500 jobs, 500 nodes, and practically no limit on the number of resource categories or length of the project in days can be accommodated [26].

While massive efforts have been expended in developing heuristic scheduling algorithms, very little effort has been expended in measuring the relative effectiveness of the heuristics which they employ. (Comparative studies which have appeared in the open literature are examined in the next section). The present investigation is an attempt to assess statistically the scheduling ability of various heuristic extensions to the critical path method in order to determine which ones are most likely to develop improved schedules. Actual multiproject data are used in this assessment, and a full factorial

*Small sets of moderate size projects refer to problems containing no more than four projects of 30 or less activities each, with each activity demanding less than five different resource types.

experiment is constructed to assess the effects of different project scheduling heuristics on various measures of organizational performance to be described.

REVIEW OF COMPARATIVE STUDIES

One of the first attempts at describing the comparative performance of selected heuristic extensions to the critical path method was performed by Brand, Meyer, and Shaffer in 1964 [2]. They solved one multiresource test problem using computerized versions of Perk's "MSP" [20], Shirley's "CPMS" [23], and Meyer and Shaffer's "RSM" [22]. They concluded that Perk's MSP was the best of the three tested, giving preference to the information printed out by each method as well as to the minimum duration schedule produced.

Fendley [7] in 1968 reported on a study performed to assess the effectiveness of selected project scheduling heuristics in satisfying eight different measures of organizational performance. Eight mock projects were formulated with each project having up to 20 activities and each activity requiring 0, 1, 2, or 3 units each of three scarce resources. Two and five project combinations of the eight hypothetical projects were then selected and scheduled with eight different heuristics (Shortest Imminent Operation, Least Total Float, etc.). Two hundred iterations of the experiment were performed. Fendley concluded that "the quantitative results did not unanimously confirm any one priority rule as best for all resource availability levels in any performance category," although the Least Total Float rule (minimum-slack-first) ranked first by four of the criteria selected and hence was judged most effective overall.

Knight [11] examined the multiproject scheduling problem under a restrictive assumption that did not allow for parallel activities in a project network. Knight's experimental medium was six sets of 10 or 11 projects consisting of two to seven activities per project. Four different resource categories were used, and each project set was scheduled using four different heuristics (Shortest Project Next, Longest Project Next, Resource Utilization Ranking, etc.). Knight's objective was to minimize makespan, the time required to complete all projects. His general conclusion was that "rules based on resource usage by each project are superior to those based on project length alone."

Mize [15] examined the job shop sequencing problem in a multiproject format. Eight hypothetical multiproject organizations were created involving three, four, and six project sets. The largest organization created involved 133 jobs (activities) and 20 departments. Under the assumptions that each job could demand the services of only one department and that each department could be at work on only one job at a time, Mize scheduled each organization using 12 different heuristics—three single-attribute heuristics and nine multiple-attribute heuristics of his own devising. Using the objective of "minimum project slippage," Mize concluded that three of the nine multiple-attribute decision rules generally yielded the best results of those tested. Each of these rules was some variation of the Least Total Float heuristic.

Finally, Pascoe [18] was the first to suggest a classification scheme for project network parameters (complexity, density, resource obstruction, etc.) and then to attempt to isolate the effect of these parameters on the performance of project scheduling heuristics. His approach was to generate 32, 20-activity networks with each activity requiring up to three resource types. Each of the generated networks satisfied specified values of his project parameters. In total, 10 "common" heuristics (Least Total Float, Late Finish Time, Late Start Time, etc.) were used to schedule each of the 32 projects. Analysis of variance procedures (in the form of a factorial experiment) were used to analyze the results. Unfortunately, the results of the experiment did not contradict a null hypothesis that there is

no significant difference between results produced by different heuristics for the majority of the performance measures examined. Pascoe did conclude, however, that the heuristics of increasing Late Finish Time or increasing Late Start Time produce the best results overall.

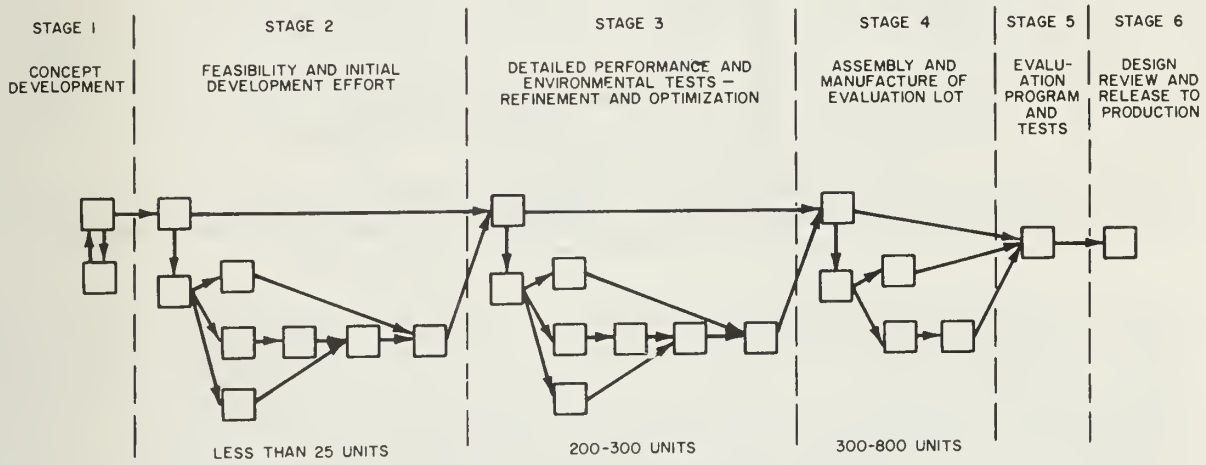
Thus there exists somewhat conflicting evidence as to the efficacy of many of the project scheduling heuristics which resolve the conflicts that develop between the resources demanded by an activity and those available. These inconsistencies are due partly to the nature of the data examined (the majority of the data is hypothetical) and partly also to the selection of heuristics and objective functions evaluated.

In the sections which follow, data describing an actual multiproject scheduling problem in a research and development department of a U.S. Navy installation are presented, and attempts by the installation to determine which scheduling rules have the highest probability of satisfying the measures of organizational performance desired are described. Then, in order to generalize the results of this investigation, scheduling problems appearing in the open literature are examined.

DESCRIPTION OF DATA

Data based on a scheduling problem at the Research and Development Department of the U.S. Naval Ammunition Depot, Crane, Indiana, are used to test the scheduling rules evaluated. A network diagram showing the sequence of activities of a typical project in this R&D organization is shown in Figure 1.

Each project consists of approximately six distinct stages; the number of stages fluctuates because of prior developmental work in a component item. Within each stage, many items are manufactured and



RESOURCES AVAILABLE TYPE	QUANTITY
ENGINEERS	21
CHEMISTS AND PHYSICISTS	14
TECHNICIANS	20
* MACHINISTS	8
ASSEMBLERS	30
TOOL DESIGNERS	6
PROCURERS	4
* PERFORMANCE EVALUATORS	10
ELECTRONIC EQUIPMENT DESIGNERS	6

* ADDITIONAL RESOURCES AVAILABLE - NON R&D

FIGURE 1. R & D Project flow.

then tested. Depending upon the results of this testing, the entire stage is either repeated or the product moves to the next stage for further developmental work. In the final stage, a report is issued either to release the item for production, or to recommend that it not be produced.

Figures 2 and 3 and Tables 1 and 2 show the distributions of selected network and resource characteristics for the problem considered. These 34 projects represent the work performed by this organization over a 10-month period, the length of time for which data are available. If a total project cannot be technologically completed within a 10-month interval, only the portion which can be completed is considered.

The project characteristics shown in Figures 2 and 3 and Tables 1 and 2 differ somewhat from those described in research efforts in which projects were generated on a computer [4], [10], [18]. For example, other researchers characteristically assume that three different resource types or groups are the maximum that can be required to complete any one activity. From Figure 2, as many as 7 of

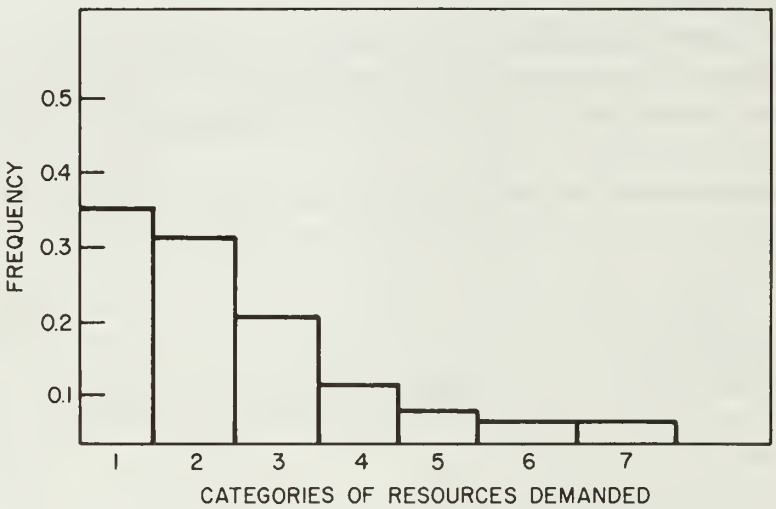


FIGURE 2. Distribution of categories of different resources demanded by each activity (each activity could demand resources from as many as 13 different resource categories).

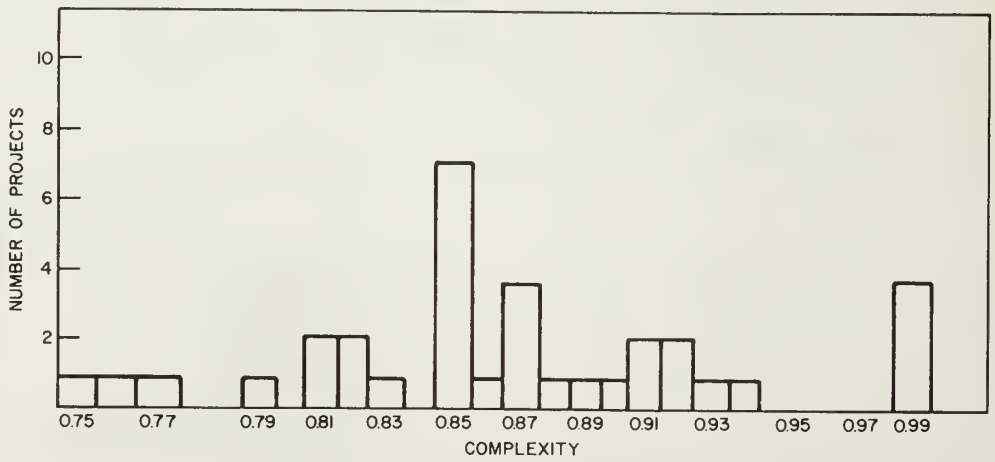


FIGURE 3. Distribution of project complexity (complexity is equal to the number of nodes, or jobs (N) in a project divided by the number of arcs, or precedence relations (M). In the related job shop problem, the complexity is equal to $N/(N-1)$.

TABLE 1. *Characteristics of Projects Scheduled*

Project number	Total man-days required	Critical path length (in days)
1	1,125	136
2	728	105
3	639	90
4	365	54
5	153	91
6	89	54
7	317	101
8	447	99
9	129	93
10	393	90
11	393	96
12	111	72
13	297	45
14	512	81
15	255	70
16	2,278	153
17	280	70
18	355	81
19	356	104
20	113	55
21	109	46
22	402	76
23	274	52
24	527	96
25	201	42
26	2,315	159
27	341	97
28	382	113
29	965	105
30	382	156
31	259	80
32	151	65
33	717	134
34	462	103

13 possible resource groups can be required to accomplish an activity. Also, the size of the projects scheduled varies substantially in man-days required to complete each project, while other simulation studies have assumed fairly "constant" project size (as measured in resource requirements). Tables 1 and 2 and Figures 1, 2, and 3 are included because field data suitable for comparing heuristic scheduling methods are generally not available, either because they are proprietary or because resource constraints are not explicitly considered and conventional CPM techniques are used.

SCHEDULING RULES EXAMINED

A variety of heuristic scheduling rules for multiproject problems can be found in the literature. The studies of Fendley, Knight, Mize, Pascoe, etc. give only a sampling of the number available. In general, the origin of the scheduling heuristics in use lies in previous research efforts in the job shop scheduling area and in the results of applications of the definitions indigenous to the critical path method, such as Late Start Time, or Late Finish Time, etc. An attempt was made to categorize the available heuristic scheduling rules so that samples from each classification could be selected and

TABLE 2. *Resource Statistics for Projects Scheduled*

Resource number	Total required man-hours	Total available direct project man-hours	Approximate resource utilization [a]	Average number of resource units demanded when required by an activity
1	979	1,260	0.78	1.65
2	2,460	3,060	0.80	2.28
3	2,918	3,600	0.81	2.72
4	699	1,080	0.65	1.28
5	2,456	3,060	0.80	1.67
6	761	1,080	0.70	1.49
7	322	1,260	0.26	1.03
8	298	720	0.41	1.18
9	3,917	5,040	0.78	1.48
10	1,179	1,620	0.73	1.18
11	286	540	0.53	1.11
12	273	540	0.51	1.54
13	123	360	0.32	1.57

[a] Because of the method used to obtain replications of the experiment (by time-phasing each of the individual projects), the actual resource utilization will differ from these figures by a small amount.

applied to the multiproject scheduling problem, the classification scheme giving some assurance that the population of available heuristics had been adequately sampled. These attempts led to the following conclusions:

1. Numerous project scheduling rules are based upon a well-defined objective, such as minimizing the completion time of the projects or maximizing the utilization of employed resources.

2. Other scheduling rules are not related specifically to an objective function, but rather reflect a characteristic of the project set or the activities comprising the project set. For example, *longest project next* assigns resources to the activity of the project having the longest expected span; any remaining activities are scheduled only if resources permit.

3. Still other scheduling rules do not relate to a specific objective function or to a characteristic of a project set, but rather use other scheduling rules as a basis on which to build. *Switch activities at random in the order in which they are considered for resource assignment when they tie on another priority* is an example of such a scheduling rule.

4. Finally, a few scheduling rules base their rationale on learning from knowledge gained in previous scheduling efforts, and are applied only after another scheduling rule has been used. *Reschedule based on resource conflicts exhibited in the first feasible schedule* is an example of such a rule.

Based on the above classification and on preliminary computational experience, the heuristic scheduling rules listed below were selected for evaluation. All rules are applied in conjunction with the parallel method of scheduling in which activity priority is determined during scheduling rather than before. All heuristic rules considered refer to the choosing of one activity over another for resource assignment because of a difference in a priority or other index associated with that activity.

1. Least Total Float (LTF)

Schedule the activities on the basis of total float, those activities possessing the least total float being considered first. The total float present in an activity is the difference between the late and early start times determined through conventional critical path analysis.

2. Greatest Total Resource Demand (GTRD)

Schedule the activities on the basis of the total resources required by the activity, that activity with the greatest resource usage being scheduled first. The rationale behind this rule lies in scheduling potential bottleneck activities because of the high usage of varied resources the activity consumes.

3. Greatest Remaining Resource Demand (GRRD)

Schedule the activities on the basis of the total remaining man-hours (resources) of work required on the project, those with the greatest remaining man-hours being scheduled first. This rule is related to many of the *look ahead* rules receiving attention in the literature today. It attempts to look ahead further than the current activity in locating potential bottlenecks; it is an extension of GTRD and changes values dynamically throughout the development of a schedule.

4. Shortest Imminent Operation (SIO)

Schedule the activities on the basis of the duration of the activity, that activity with the shortest duration being scheduled first. This is analogous to the SIO rule of the job shop.

5. Greatest Resource Utilization (GRU)

Schedule the activities at time t to maximize the utilization of available man-power (resources). This rule is formulated and solved as a zero-one integer programming problem where $x_j = 1$ if activity $j(a_j)$ is to be scheduled during the current time interval, and 0 if it is not:

$$\max \sum_j c_j x_j$$

$$\text{subject to } R^t \underline{x} \leq \underline{b}^t$$

$$x_j \in \{0, 1\}$$

b_i^t = resource i available at time t; b^t dimension $q \times 1$

$$c_j = \sum_{i=1}^{q} r_{ij}$$

r_{ij} = resource i required by a_j

R^t = matrix of resources required by all a_j which can be scheduled at time t.

Note that r_{ij} can be defined for man-power, machines, etc., but that when different types of resources are scheduled simultaneously, it becomes necessary to define common units for them. For example, if both men and machines are to be scheduled at time t, a value (cost) can be associated with each, in which case one maximizes the "value" of employed resources. For the data described, r_{ij} refers to man-power, but ordinarily would not have to be restricted to this type of resource.

The application of the Greatest Resource Utilization rule demonstrates that heuristic scheduling rules are not always based on priority indices for activities and subsequently cannot be implemented solely through sorting routines.

6. Randomness (RAN)

If any activities which are candidates for scheduling during the current time interval tie on any given priority, switch them at random in the order in which they are considered for resource assignment. The randomness rule is never applied by itself. Despite the simplicity of this rule, attempts [26] to solve certain project scheduling problems without it have been unsuccessful.

7. Reschedule (RES)

Determine which class or category of resource had the least total amount of idle time in the first feasible schedule. Develop a new schedule using as activity priority the quantity of this resource required, multiplied by the duration of the activity.

Other research efforts [11], [15] have determined that improvements can be made in schedules either by rescheduling on the basis of a resource that caused more delays in activities in the first schedule, or by including provisions for gap-closing by rescheduling those activities that are scheduled to start before their technological (CPM) late start time. The reschedule feature used here is representative of the former reschedule routine.

TABLE 3. *Scheduling Rules Examined*

Scheduling rule	Identification	Basis for inclusion
Least total float	LTF	Take advantage of the slack time present in an activity.
Greatest total resource demand	GTRD	Schedule potential bottleneck activities.
Greatest remaining resource demand	GRRD	Look ahead further than the current activity in scheduling potential bottleneck activities.
Shortest imminent operation	SIO	Process as many jobs through the system as rapidly as possible in an attempt to minimize delays.
Greatest resource usage	GRU	Schedule all resources that can be scheduled during a time interval.
Randomness	RAN	Add an element of chance in the assignment of resources to an activity.
Reschedule	RES	Reschedule activities on the basis of critical or tight resources exhibited in the first feasible schedule.

PERFORMANCE CRITERIA

The following criterion functions are used to assess the efficacy of the scheduling rules and scheduling programs (combinations of scheduling rules) examined. The choice of an appropriate objective function may differ in various scheduling environments and in different periods of time. Several of the common ones are therefore selected for examination.

1. Total Project Delay

The total delay of a project set is the sum (over all projects) of the difference between the assigned scheduled finish time of a project and the length of the critical path in an early start schedule. It is doubtful that many of the projects considered in a resource-constrained, multiproject scheduling problem will be completed within the critical path completion time estimate. This measure does, however, give an indication of the delays introduced as a result of limitations on resource availability and as a result of the scheduling rule employed.

2. Weighted Total Delay

The weighted total delay of a project set is the sum (again, over all projects) of the total resources demanded by a project multiplied by the total delay of the project as defined above. Weighted delay is measured in man-days for the data described. This measure places additional emphasis on the seriousness of a delay in the larger projects.

3. Total Resource Idle Time

The total resource idle time is the amount of time that resources are idle during a schedule span. Idle time is measured in man-hours; it is a result of the unavailability of direct project work, which in turn is a result of the scheduling method employed.

In order to avoid biasing this summary measure by using different time intervals for measuring idleness, a measure of the idle resources up to and including 140 days after the start date for scheduling (day 0) was used. The limit of 140 days for accumulating idle resources represents approximately three-fourths of the length of the schedule span generated by each heuristic scheduling procedure.

4. Computer Processing Time

The computer processing time is the amount of time expended in generating a schedule. It is measured in seconds and is an indication of the direct cost of using the scheduling rule chosen.

EXPERIMENTAL DESIGN

A randomized complete block, full factorial design is used to assess the efficacy of the heuristic scheduling rules examined. The blocking performed is achieved by varying the starting times of each of the 34 projects. These starting times are generated by drawing random variates which are uniformly distributed over a span of 180 days, the length of a schedule. For each block, a string of 34 random variates is drawn. The first random variate is the starting time of the first project, the second is the starting time of the second project, etc. One string of 34 random variates corresponds to one block of the experiment. This method of obtaining replications of the experiment has the effect of varying the total resources required during the schedule span, and hence affects the resource usage rate.

Of the seven scheduling rules examined, five (LTF, GTRD, GRRD, SIO, GRU) are applied to a project scheduling problem independently of one another. These five main scheduling rules are called treatment A in the discussion which follows.

The remaining two scheduling rules (RAN & RES) are applied only in conjunction with one of the first five rules. They are labeled treatments B and C, respectively. These last two treatments exist at one of two possible levels; they are either present in a heuristic program or they are not. (This is to be distinguished from treatment A, where one level of the factor must always be present in order to develop a schedule.)

Thus, one factor (treatment A) is varied over five levels and each of two factors (treatments B and C) are varied over two levels. The experiment is then a $5 \times 2 \times 2$ factorial experiment; from seven scheduling heuristics, 20 ($5 \times 2 \times 2$) treatment combinations are formed. All 20 treatment combinations are investigated.

For each of the four descriptive measures evaluated, a four-way and a two-way analysis of variance is given in order to assess interactions, main effects, and effects due to blocking or introducing different resource demands. Duncan's Multiple Range Test [6] is then used to rank the various means; the 5-percent level of significance is used for reporting significant differences.

A total of 30 replications (blocks) of the experiment are made. This number is based on previous multiproject scheduling research [15] and the desire to reject the hypothesis with the power at 0.95 and $\alpha = 0.05$ that the scheduling rules are equal when in fact one of them exceeds the others by σ, the population standard deviation. Examination of power curves of the Non-Central F-Distribution developed by Pearson and Hartley [8], [19] reveals that a sample of $n = 30$ is sufficient to insure the accuracy stated.

TEST RESULTS

A computer program called MPSP, an acronym for Multi-Project Scheduling Program,* was written to implement the scheduling rules described. The program was run on the Indiana University CDC 3600 computer. The computer processing times herein reported refer to average time spent in generating a schedule, exclusive of all *I/O* time.

Bartlett's [5], Cochran's [5], and Hartley's Short-Cut [9] test for the homogeneity of variances are made for all of the summary measures. With Bartlett's test, the hypothesis that the variances associated with total project delay and computer processing time are homogeneous is rejected at the 5-percent level of significance; the hypothesis for the measures weighted total delay and total resource idle time is accepted. With Cochran's and Hartley's tests, the variances associated with each summary measure are concluded to be homogeneous at the 5-percent level of significance. The variances are therefore treated as being homogeneous. These results are summarized in Table 4.

Having concluded that the variances are homogeneous, the statistical results of ranking each of the scheduling rules are now given.† An attempt is also made to state which scheduling rule is superior for each of the criterion functions examined.

1. Total Project Delay

The average amount of total project delay varies between 998 and 1,502 days depending upon the scheduling rule used. The application of the Shortest Imminent Operation heuristic produces the least total project delay; the heuristic Greatest Remaining Resource Demand produces the highest amount of delay.

TABLE 4. *Tests for Homogeneity of Variances*

Criterion function	Bartlett's test computed value	Cochran's test computed value	Hartley's test computed value
Total project delay	[a] 1.69	0.089	3.82
Weighted total delay	1.01	0.082	3.07
Total resource idle time	0.98	0.092	3.78
Computer processing time	[a] 1.81	0.091	3.65

[a] The variances are concluded to be homogeneous at the 5-percent significance level with two exceptions. The hypothesis of homogeneous variances is rejected at the 5-percent level using Bartlett's test for the measures of total project delay and computer processing time. The hypothesis of homogeneous variances is accepted using each test and for all of the summary measures at the 1-percent significance level.

*MPSP was written by J. A. Werne of NAD Crane.

†The results herein reported are coded, since they are not intended for public use. This coding does not affect the analysis or the interpretation of the results.

The presence of Randomness (Treatment B in Table 5) when used in conjunction with the other scheduling heuristics is not generally expected to produce better results; it might on the average even be disfunctional. But occasionally, the presence of some element of chance in the selection of activities for resource assignment may lead to optimal solutions to project scheduling problems. Thus, as might be expected, randomness contributes little in explaining the variation present in total project delay as shown in Table 5.

It could conceivably be argued that the statistical analysis is not particularly appropriate when the factor (treatment) for randomness is included in the analysis. However, for completeness, it is included. If randomness were significant in the analysis of variance, this would demonstrate that the heuristics being examined did not discriminate amongst the activities. For example, if all of the durations of the activities were equal, the Shortest Imminent Operation heuristic would in essence be a "Select Activities at Random" rule. In this instance, the randomness rule could be construed to be another heuristic which should be included in Treatment *A*. Thus there is some information in the fact that Treatment *B* is insignificant, albeit minor.

The main scheduling rule and the reschedule routine contribute significantly in explaining the variation present in a schedule. The presence of rescheduling, however, generally increases the amount of total project delay present. Only for the rules Greatest Remaining Resource Demand and Greatest Total Resource Demand is total project delay reduced by the presence of the reschedule rule. Both of these rules, however, represent the worst instances of total project delay.

TABLE 5. *Analysis of Variance for Total Project Delay*

Source	*d.f.*	*S.S.*	*M.S.*	*F*	$F_{\alpha=0.01}$
A	4	6,491,757	1,622,939	386.38	3.35
B	1	6,767	6,767	1.61	6.68
C	1	279,288	279,288	66.49	6.68
AB	4	10,318	2,580	0.61	3.35
AC	4	4,394,954	1,098,738	261.58	3.35
BC	1	5,287	5,287	1.26	6.68
ABC	4	6,455	1,614	0.38	3.35
Blocks	29	1,761,246	60,732	14.46	1.75
Error	551	2,314,408	4,200		
Total	559	15,270,481			

TABLE 6. *Analysis of Variance for Total Project Delay Main Scheduling Heuristic Only*

Source	*d.f.*	*S.S.*	*M.S.*	*F*	$F_{\alpha=0.01}$
Treatments	4	5,126,712	1,281,678	294.62	3.48
Blocks	29	518,316	17,873	4.11	1.86
Error	116	504,627	4,350		
Total	149	6,149,655			

The results of applying Duncan's test to the observed data are shown in Table 7. The Shortest Imminent Operation heuristic produces the lowest ranking sample mean.

TABLE 7. *Multiple Range Test for Total Project Delay*

Mean (in days)	1,502	1,487	1,351	1,342	1,206	1,201	1,161	1,155	1,154	1,151	1,150	1,138	1,132	1,131	1,121	1,110	1,054	1,043	1,000	998
Treatment	GRRD RAN	GRRD	GTRD RAN	GTRD	GRRD RAN RES	GRRD RES	GTRD RAN RES	GRU RES	GRU RAN RES	LTF RAN RES	SIO RAN RES	SIO RES	GTRD RES	LTF RES	GRU	GRU RAN	LTF	LTF RAN	SIO RAN	SIO

A horizontal line enclosing a group of means indicates that the means located within the group cannot be distinguished from one another at the 5-percent level of significance. Descriptions of each of these treatments are found in Table 3.

2. Weighted Total Delay

The scheduling heuristic which produces the least weighted total delay is Least Total Float. Duncan's multiple range test is, however, unable to distinguish between this scheduling rule and the Greatest Resource Usage rule. Since the Least Total Float rule requires approximately one-half of the computer processing time that the Greatest Resource Usage rule requires, Least Total Float is the better scheduling rule to use to minimize weighted total delay. Results of applying Duncan's test are shown in Table 10.

TABLE 8. *Analysis of Variance for Weighted Total Delay*

Source	*d.f.*	*S.S.* (100,000)	*M.S.* (100,000)	*F*	$F_{\alpha=0.01}$
A	4	192,859	48,215	32.95	3.35
B	1	1,699	1,699	1.16	6.68
C	1	19,963	19,963	13.64	6.68
AB	4	4,022	1,005	0.69	3.35
AC	4	130,052	32,513	22.22	3.35
BC	1	953	953	0.65	6.68
ABC	4	3,218	805	0.55	3.35
Blocks	29	417,217	14,387	9.83	1.75
Error	551	806,356	1,463		
Total	599	1,223,573			

TABLE 9. *Analysis of Variance for Weighted Total Delay Main Scheduling Heuristic Only*

Source	*d.f.*	*S.S.* (100,000)	*M.S.* (100,000)	*F*	$F_{\alpha=0.01}$
Treatments	4	180,315	45,079	22.50	3.48
Blocks	29	117,411	4,049	2.02	1.86
Error	116	232,388	2,003		
Total	149	530,114			

TABLE 10. *Multiple Range Test for Weighted Total Delay*

Mean (in man-days)	723,693	723,395	723,127	721,692	719,803	714,011	713,060	710,927	703,471	703,235	702,360	698,327	697,270	695,129	693,777	692,352	660,190	655,869	653,746	644,648
Treatment	GRRD RAN RES	GTRD	GRRD RES	SIO	GRRD RAN	GTRD RAN	GRRD	SIO RAN	LTF RAN RES	GRU RAN RES	GTRD RAN RES	SIO RAN RES	GRU RES	SIO RES	GTRD RES	LTF RES	GRU RAN	LTF RAN	GRU	LTF

A horizontal line enclosing a group of means indicates that the means located within the group cannot be distinguished from one another at the 5-percent level of significance. Descriptions of each of these treatments are found in Table 3.

The four-way analysis of variance given in Table 8 shows treatment A, treatment C (the reschedule feature), and the A–C treatment interaction are significant beyond the 1-percent level of significance in the analysis of variance.

3. Total Resource Idle Time

The scheduling abilities of the Shortest Imminent Operation heuristic and the Greatest Remaining Resource Demand rule are reversed when measuring total resource idle time as opposed to total project delay. The Greatest Remaining Resource Demand rule produces schedules with the least total resource idle time, and the Shortest Imminent Operation heuristic produces schedules with the greatest amout of resource idle time. Such a result can be expected. The logic behind the Greatest Remaining Resource Demand rule lies in scheduling potential bottleneck activities and hence in utilizing resources efficiently, while the logic of the Shortest Imminent Operation heuristic lies in accomplishing as many jobs as possible in as short a length of time as possible. The results of applying Duncan's test to the criterion total resource idle time are shown in Table 13.

The four-way analysis of variance for total resource idle time is shown in Table 11. As with the previous measures for assessing scheduling ability, treatments A and C and the A–C treatment interaction are highly significant in explaining the variation present in schedules, whereas treatment B, randomness, is insignificant.

TABLE 11. *Analysis of Variance for Total Resource Idle Time*

Source	*d.f.*	*S.S.*	*M.S.*	*F*	$F_{\alpha=0.01}$
A	4	971,843,896	242,960,974	78.84	3.35
B	1	168,371	168,371	0.05	6.68
C	1	579,515,883	579,515,883	88.06	6.68
AB	4	5,948,609	1,487,152	0.48	3.35
AC	4	639,991,514	159,997,879	51.92	3.35
BC	1	12,595	12,595	0.00	6.68
ABC	4	8,764,936	2,191,234	0.71	3.35
Blocks	29	1,011,357,775	34,874,405	11.32	1.75
Error	551	1,697,964,539	3,081,605		
Total	599	4,915,568,097			

TABLE 12. *Analysis of Variance for Total Resource Idle Time Main Scheduling Heuristic Only*

Source	*d.f.*	*S.S.*	*M.S.*	*F*	$F_{\alpha=0.01}$
Treatments	4	790,870,250	197,717,563	59.15	3.48
Blocks	29	274,677,502	9,471,638	2.83	1.86
Error	116	387,737,532	3,342,565		
Total	149	1,453,285,284			

TABLE 13. *Multiple Range Test for Total Resource Idle Time*

Mean (in man-hours)	68,203	67,694	66,583	66,571	66,557	66,502	56,484	66,419	66,414	66,394	65,732	65,681	65,599	65,067	64,080	63,987	63,595	63,364	61,193	60,736
Treatment	SIO	SIO RAN	GRU RAN RES	SIO RAN RES	LTF RES	GTRD RES	GRU RES	GTRD RAN RES	SIO RES	LTF RAN RES	GRRD RES	LTF RAN	GRRD RAN RES	LTF	GTRD	GTRD RAN	GRU RAN	GRU	GRRD	GRRD RAN

A horizontal line enclosing a group of means indicates that the means located within the group cannot be distinguished from one another at the 5-percent level of significance. Descriptions of each of these treatments are found in Table 3.

4. Computer Processing Time

The computer processing time criterion is one measure of the cost of operating a scheduling system. Table 16 shows the average time (in seconds) required to develop a schedule. As can be seen, the six lowest ranking sample means differ at the most by only 3 sec (CDC 3600 CPU time). And while the sixteen lowest ranking sample means differ by as much as 46 sec, this difference is of little economic significance. Computer processing time is therefore concluded to be of little economic significance in developing heuristic schedules for the scheduling problem considered.

TABLE 14. *Analysis of Variance for Computer Processing Time*

Source	*d.f.*	*S.S.*	*M.S.*	*F*	$F_{\alpha=0.01}$
A	4	608,183	152,046	8225.61	3.35
B	1	648	648	35.08	6.68
C	1	54,037	54,037	2923.39	6.68
AB	4	394	98	5.33	3.35
AC	4	154	39	2.09	3.35
BC	1	30	30	1.65	6.68
ABC	4	0	0	0	3.35
Blocks	29	2,176	75	4.06	1.75
Error	551	10,185	18		
Total	599	675,808			

TABLE 15. *Analysis of Variance for Computer Processing Time Main Scheduling Heuristic Only*

Source	*d.f.*	*S.S.*	*M.S.*	*F*	$F_{\alpha=0.01}$
Treatments.....	4	146,386	36,596	1396.65	3.48
Blocks.....	29	811	28	1.07	1.86
Error.....	116	3,040	26		
Total.....	149	150,237			

TABLE 16. *Multiple Range Test for Computer Processing Time*

Mean (in seconds)	126	120	107	102	66	65	48	47	44	42	42	40	40	38	23	22	22	22	21	20
Treatment	GRU RAN RES	GRU RES	GRU RAN	GRU	LTF RAN RES	LTF RES	LTF RAN	LTF	GRRD RAN RES	GRRD RES	GTRD RAN RES	GTRD RES	SIO RAN RES	SIO RES	GTRD RAN	GRRD RAN	GTRD	GRRD	SIO RAN	SIO

A horizontal line enclosing a group of means indicates that the means located within the group cannot be distinguished from one another at the 5-percent level of significance. Descriptions of each of these treatments are found in Table 3.

PROJECT SCHEDULING EXPERIENCE: HYPOTHETICAL DATA

Six sets of fictitious projects ranging in size from four to six projects each were adapted from [15] and scheduled with MPSP. These data differ substantially from that previously analyzed in that each activity requires only one resource type, and each of the 20 available resource types can be at work on no more than one activity at a time. This then is actually job shop sequencing data viewed in a multiproject format. The analysis of this data presents a different type of challenge to the scheduling rules than does the data previously examined.

These project sets were scheduled with each of the six sets being considered a different replicate of the experiment. Table 17 presents the analysis of variance results for the criterion total project delay. The notation is the same as used in the previous section, with treatment A being the main scheduling heuristics, treatment B being randomness, etc.

The F-Ratio for replications shown in Table 17 is far greater ($414.8 > 10.5$) than the F-Ratio for the main scheduling heuristics. This indicates that for the hypothetical data considered, project and resource characteristics contribute more in explaining the variation present in project schedules than do the scheduling methods employed.

Analysis of variance results for each of the remaining summary measures produced similar results and hence are not shown; the F-Ratio for replicates is easily 20 times as large as the F-Ratio for any of the remaining sources of variation. No reason is given in the reference as to why the particular structure was chosen for the project sets producing these results, but results similar to the above have been noted when analyzing the single project problem using hypothetical data [18].

TABLE 17. *Analysis of Variance for Total Project Delay: Hypothetical Data*

Source	*d.f.*	*S.S.*	*M.S.*	*F*
A	4	72,141	18,035	10.52
B	1	20	20	0.01
C	1	29,954	29,954	17.47
AB	4	333	83	0.05
AC	4	53,367	13,341	7.78
BC	1	469	469	0.27
ABC	4	54	14	0.01
Replications	5	3,556,728	711,345	414.84
Error	95	162,902	1,715	
Total	119			

Since the *F*-Ratio for replicates is so large, no attempt was made to rank the scheduling heuristics using multiple ranking procedures. The sample means for the main scheduling heuristics are, however, shown in Table 18 omitting the statistical analysis. The ordinal ranking of many of the heuristic scheduling rules is similar to that noted in examining real data.

TABLE 18. *Sample Means for Main Scheduling Heuristic: Hypothetical Data*

Scheduling heuristic	Performance criteria			
	Total project delay [a]	Total weighted delay [a]	Total resource idle time [a]	Computer processing time [b] (sec)
Least total float	421	260,155	4,802	3.59
Greatest total resource demand	482	302,830	5,097	1.95
Greatest remaining resource demand	484	279,857	4,447	2.13
Shortest imminent operation	438	275,166	5,007	1.99
Greatest resource usage	460	286,158	4,892	7.02

[a] Since hypothetical data are used, no units are attached to these means.
[b] CDC 3600 CPU Time.

An attempt was also made to schedule a single project problem posed by Martino [14]. This problem was fabricated to produce bad results using many of the heuristic scheduling rules available. With a resource limit of eight men and without using the random switching feature, MPSP is unable to produce an optimal* schedule for this problem. With the random switching feature turned on, however, the algorithm found an optimal schedule using the Greatest Resource Usage heuristic. The algorithm was also able to identify five different optimal schedules when it was allowed to repeat the Greatest Resource Usage rule with randomness a total of 20 times. These results are similar to those reported by Wiest [26].

CONCLUSION

Optimal solutions to the constrained-resource, project scheduling problem are infeasible at the present for all but moderate size problems. And while heuristic approaches offer the most promising results to date, several heuristic approaches should be examined and compared to the current require-

*The critical path length for this project is 19 days and the project can be accomplished with as few as eight men in a 19-day period. Because the project can be completed in nineteen days with no variation in resource usage and no idle resource time, a 19-day schedule is considered optimal.

ments of a good schedule before any one of them is selected. The results reported demonstrate that it is economically practical to employ several heuristic scheduling rules and to then choose a schedule which comes closest to meeting desired objectives.

The results of examining actual and laboratory type projects suggest that the scheduling rules which produce schedules with low total project delays do so at the expense of an inefficient utilization of resources; the scheduling methods which schedule resources efficiently do so at the expense of large delays in project completions. While it was not possible to distinguish among heuristic scheduling rules when hypothetical data were examined, the results reported demonstrate that when actual project data are considered, the scheduling methods employed account for a significant portion of the variation present in project schedules.

It has been suggested that the evaluation of heuristic models should be based on their usefulness rather than on their ability to obtain optimum solutions to problems. While the Multi-Project Scheduling program was able to identify five optimum solutions to a specific problem, in general it cannot be claimed that an optimum is obtainable by these methods.

Field data suitable for comparing heuristic scheduling methods are generally not available either because they are proprietary or because resource constraints are not explicitly considered and conventional CPM techniques are used. The inclusion of an actual multiproject, constrained-resource scheduling problem should aid researchers in the area in comparing other heuristic scheduling algorithms and in selecting characteristics for generating laboratory projects.

BIBLIOGRAPHY

[1] Bennett, Fred L., "Some Approaches to the Critical Path Scheduling Resource Allocation Problem," unpublished Ph.D. Thesis, Cornell University (1966).

[2] Brand, J. D., W. L. Meyer, and L. R. Schaeffer, "The Resource Scheduling Problem in Construction," Civil Engineering Studies, Report No. 5, Urbana, Illinois: Department of Civil Engineering, University of Illinois (1964).

[3] Davies, Owen L. (ed.), *The Design and Analysis of Industrial Experiments* (Oliver and Boyd, London, 1960).

[4] Davis, E. W., "An Exact Algorithm for the Multiple Constrained-Resource Project Scheduling Problem," unpublished Ph.D. Thesis, Yale University (1969).

[5] Dixon, Wilfrid J. and Frank J. Massey, Jr., *Introduction to Statistical Analysis* (McGraw-Hill Book Company, Inc., New York, 1969).

[6] Duncan, Acheson J., *Quality Control and Industrial Statistics* (Rev. Ed.) (Richard D. Irwin, Inc., Homewood, Illinois, 1959).

[7] Fendley, L. G., "Toward the Development of a Complete Multiproject Scheduling System," The Journal of Industrial Engineering, Vol. 19, No. 10 (Oct. 1968), pp. 505–515.

[8] Guenther, William C., *Analysis of Variance.* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964).

[9] Hartley, H. O., "The Maximum F-Ratio as a Short-Cut Test for Heterogeneity of Variance," Biometrica, Vol. 37 (1950), pp. 308–312.

[10] Johnson, Thomas J. R., *An Algorithm for the Resource-Constrained, Project Scheduling Problem.* Unpublished Ph.D. Thesis, Massachusetts Institute of Technology (1967).

[11] Knight, R. M., "Resource Allocation and Multi-Project Scheduling in a Research and Development Environment," Unpublished M.S. Thesis, Massachusetts Institute of Technology (June 1966).

[12] Lambourn, S., "RAMPS—A New Tool in Planning and Control," The Computer Journal, Vol. 5 (1963), pp. 300–304.

[13] Levy, F. K., G. L. Thompson, and J. D. Wiest, "Multi-Ship, Multi-Shop Workload—Smoothing Programs," Nav. Res. Log. Quart. *9*, 37–45 (Mar. 1962).

[14] Martino, R. L., Project Management and Control: Vol. 111, Allocating and Scheduling Resources, (American Management Association, New York, 1965).

[15] Mize, J. H., "A Heuristic Scheduling Model for Multi-Project Organizations," unpublished Ph.D. Thesis, Purdue University (1964).

[16] Moshman, J., J. Johnson, and M. Larsen, "RAMPS—A Technique for Resource Allocation and Multiproject Scheduling," Proceedings—1963 Spring Joint Computer Conference, pp. 17–27.

[17] Muth, John F. and Gerald L. Thompson (Eds.), *Industrial Scheduling* (Prentice-Hall, Inc., New Jersey, 1963).

[18] Pascoe, T. L., "An Experimental Comparison of Heuristic Methods for Allocating Resources," unpublished Ph.D. Thesis, Cambridge University (1965).

[19] Pearson, E. S. and H. O. Hartley, "Charts of the Power Function for Analysis of Variance Tests Derived from the Non-Central F-Distribution," Biometrica, Vol. 38 (1951), pp. 112–130.

[20] Perk, H. N., "Man-Scheduling Program for the IBM 1620," Revised, IBM Program Library File No. 10.3.013.

[21] Pritsker, A. A. B. and L. J. Watters, "A Zero-One Programming Approach to Scheduling with Limited Resources," The RAND Corporation, RM-5561-PR (Jan. 1968).

[22] Shaffer, L. R., J. B. Ritter, and W. L. Meyer, *The Critical Path Method* (McGraw-Hill, New York, 1965).

[23] Shirley, W. W. and I. L. Bowman, "8,000, Critical Path and Man Scheduling," Richfield Oil Co., Los Angeles, California (1962).

[24] Tonge, Fred, "The Use of Heuristic Programming in Management Science," Management Science, Vol. 7, No. 1 (Oct. 1960), pp. 21–42.

[25] Wiest, J. D., "Some Properties of Schedules for Large Projects with Limited Resources," Operations Research, Vol. 12, No. 3 (May–June 1964), pp. 395–418.

[26] Wiest, J. D., "A Heuristic Model for Scheduling Large Projects with Limited Resources," Management Science, Vol. 13, No. 6 (Feb. 1967), pp. 359–377.

[27] Wiest, J. D., "The Scheduling of Large Projects with Limited Resources," unpublished Ph.D. Thesis, Carnegie Institute of Technology (1963).

[28] Winer, B. J., *Statistical Principles in Experimental Design.* (McGraw-Hill Book Company, Inc., New York, 1962).

OPTIMUM ADJUSTMENT POLICY FOR A PRODUCT WITH TWO QUALITY CHARACTERISTICS

William J. Kennedy, Jr.
University of Utah

and

Prabakhar M. Ghare
Virginia Polytechnic Institute and State University

INTRODUCTION

We consider the problem of determining an optimal adjustment policy when the price received for the product is a function of a stated quality measure. When this quality measure has a specified value maximum price can be received. As the quality measure deviates from the specified value the price received drops progressively. An example of this type of pricing is shown in Table 1.

The quality measure is a function of two characteristics X and Y. When the process is properly adjusted the characteristics have nominal values x_0 and y_0, respectively; this results in most of the production being in the highest priced category q_3: The process would remain in this state up to the occurence of disruptive events at t_x and t_y, which would indicate the beginning of an increase in the values of X and Y, respectively. In each case, the increase is assumed to follow a known, differentiable monotonic, invertible function of the time elapsed after the disruptive event. X and Y are thus random variables with values x and y given by these equations:

$$x = x_0 + h(t - t_x) \qquad t > t_x$$

$$= x_0 \qquad 0 \leq t \leq t_x$$

$$y = y_0 + k(t - t_y) \qquad t > t_y$$

$$= y_0 \qquad 0 \leq t \leq t_x$$

An example of the behavior of X and Y is shown in Figure 1.

TABLE 1

Pricing category	Lower limit	Upper limit	Price received/unit
q_1	$L_1 = -\infty$	$U_1 = 4.7$	$R_1 = -\$0.70$
q_2	$L_2 = 4.7$	$U_2 = 4.9$	$R_2 = \$1.10$
q_3	$L_3 = 4.9$	$U_3 = 5.2$	$R_3 = \$3.00$
q_4	$L_4 = 5.2$	$U_4 = 5.6$	$R_4 = \$0.10$
q_5	$L_5 = 5.6$	$U_5 = +\infty$	$R_5 = -\$0.50$

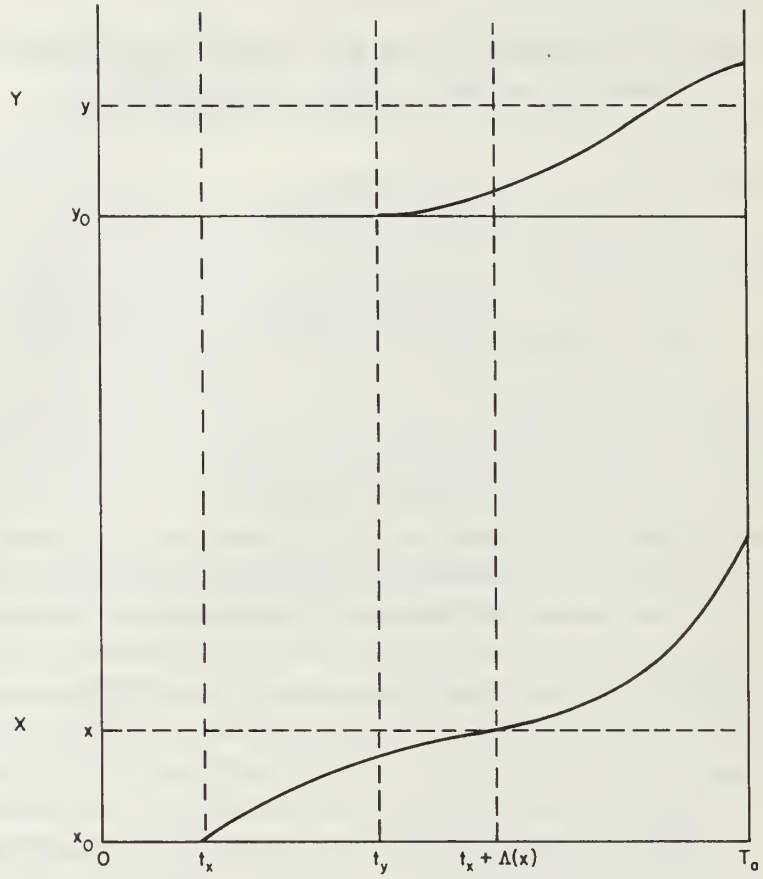


FIGURE 1. An illustration of the nomenclature used

The disruptive events are assumed to occur at random times with known probability distributions each disruptive event independent of the other. After an adjustment both characteristics return to their nominal values x_0 and y_0, respectively. The objective is to determine an optimal time interval between adjustments which would maximize the expected net revenue minus the cost of adjustments. Adjustments are assumed to be performed after equal time intervals.

Problems similar to this have been treated by Pritsker [3], Gibra [1], and Girschick and Rubin [2]. None of these authors have, however, considered explicitly the case where the quality measure comes from a probability distribution whose parameters are stochastic, nor have they considered problems where the price paid is an explicit function of the quality measure. Roeloffs [4] has presented a method of obtaining quality-determined differential pricing places for attributes sampling, but he has not used his differential pricing plans to determine adjustment intervals.

Symbols and Definitions

X = one characteristic—a random variable
Y = the other characteristic—also a random variable
x_0 = the starting, or nominal, value of X
y_0 = the starting, or nominal, value of Y

T_a = the time between successive adjustments
T_x = the time X begins to increase, a random variable
T_y = the time Y begins to increase, a random variable
$f_{T_x}(t_x)$ = the probability density function of T_x, evaluated at t_x
$f_{T_y}(t_y)$ = the probability density function of T_y, evaluated at t_y
$h(t-t_x)$ = the function describing the increase in X as a function of elapsed time since t_x
$= 0,\ t \leq t_x$
$k(t-t_y)$ = the function describing the increase in Y as a function of the elapsed time since t_y
$= 0,\ t \leq t_y$
$f_{x,y}(w, y)$ = the joint probability mass-density function of X and Y,
$A(x)$ = the value of the elapsed time since t_x for which X is equal to x
$= h^{-1}(x-x_0),\ x_0 < x \leq x_0 + h(T_a)$
$= 0,\ x \leq x_0$
$B(y)$ = the value of the elapsed time since t_y for which Y is equal to y
$= k^{-1}(y-y_0),\ y_0 < y \leq y_0 + k(T_a)$
$= 0,\ y \leq y_0$

The Joint Mass-Density Function for X and Y

Since the two characteristics X and Y are random variables, their joint mass density function can be computed by considering four mutually exclusive and exhaustive cases represented by (x_0, y_0), (x_0, y), (x, y_0), and (x, y), respectively.

$$f_{X,Y}(x_0, y_0) = \int_0^{T_a} f_{T_x}(t_x)\frac{t_x}{T_a}[1-F_{T_y}(t_x)]dt_x + \int_0^{T_a} f_{T_y}(t_y)\frac{t_y}{T_a}[1-F_{T_x}(t_y)]dt_y$$
$$+ \int_{T_a}^{\infty} f_{T_x}(t_x)[1-F_{T_y}(t_x)]dt_x + \int_{T_a}^{\infty} f_{T_y}(t_y)[1-F_{T_x}(t_y)]dt_y;$$

$$f_{X,Y}(x, y_0) = \frac{dA(x)}{dx}\int_0^{T_a-A(x)} f_{T_x}(t_x)\frac{1}{T_a}(1-F_{T_y}[t_x+A(x)]dt_x X_0 < x \leq x_0 + h(T_a);$$

$$f_{X,Y}(x_0, y) = \frac{dB(y)}{dy}\int_0^{T_a-B(y)} f_{T_y}(t_y)\frac{1}{T_a}(1-F_{T_x}[t_y+B(y)])\,dt_y,\ y_0 < y \leq y_0 + k(T_a);$$

$$f_{X,Y}(x, y) = \begin{cases} \dfrac{dA(x)}{dx}\dfrac{dB(y)}{dy}\displaystyle\int_0^{T_a-A(x)} \frac{1}{T_a} f_{T_x}(t_x) f_{T_y}[t_x+A(x)-B(y)]dt_x, \\ \qquad A(x) > B(y) \\ \qquad x_0 < x \leq x_0 + h(T_a) \\ \qquad y_0 < y \leq y_0 + k(T_a) \\ \dfrac{dA(x)}{dx}\dfrac{dB(y)}{dy}\displaystyle\int_0^{T_a-B(y)} \frac{1}{T_a} f_{T_y}(t_y) f_{T_x}[t_y+B(y)-A(x)]\,dt_y \\ \qquad A(x) \leq B(y) \\ \qquad x_0 < x \leq x_0 + h(T_a) \\ \qquad y_0 < y \leq y_0 + k(T_a). \end{cases}$$

In the production process modeled, X and Y are, respectively, the mean and standard deviation of a normal distribution from which a given measurement is assumed to come. This measurement, a random variable Z, has the cumulative distribution function given by

$$(1) \quad F_Z(z) = \int_{-\infty}^{z} \phi(w \mid x_0, y_0) f_{x,y}(x_0, y_0)\, dw + \int_{X_0}^{x_0+h(T_a)} \int_{-\infty}^{z} \phi(w \mid x, y_0) f_{x,y}(x, y_0)\, dw dx$$

$$+ \int_{y_0}^{y_0+k(T_a)} \int_{-\infty}^{z} \phi(w \mid x_0, y) f_{x,y}(x_0, y)\, dw dy$$

$$+ \int_{x_0}^{x_0+h(T_a)} \int_{y_0}^{y_0+k(T_a)} \int^{z} \phi(w \mid x, y) f_{x,y}) f_{x,y}(x, y)\, dw dx dy,$$

where $\phi(w \mid x, y) = \dfrac{1}{y\sqrt{2\pi}} \exp\left[-\dfrac{1}{2}\left(\dfrac{w-x}{y}\right)^2\right]$.

The probability $P(q_i^*)$ that an item falls into pricing category q_i is then

$$P(q_i^*) = F_Z(U_i) - F_Z(L_i).$$

The net revenue can be described as the price paid for produced items minus the cost of adjustment. The expected net revenue can be expressed as

$$(2) \quad E\,[\text{Net revenue}] = N \sum_{i=1}^{5} R_i P(q_i^*) - C_a T_R / T_a,$$

where

N = number of items produced in each production run
R_i = per unit revenue in pricing category P_i
C_a = cost per adjustment
T_R = duration of each production run
T_a = time between successive adjustments

The problem is to find T_a so that the net revenue is a maximum.

DETERMINATION OF AN OPTIMUM INTERVAL

An assumption is made on the pricing categories. This assumption is that the pricing category including the nominal value x_0 of the process mean has the highest per unit price and prices in other categories decrease as they move away from x_0, i.e.,

$$x_0 \epsilon q_i \Rightarrow R_i > R_j, j \neq i$$

and, for $j, k \neq i$,

$$|L_j - X_0| > |L_k - X_0| \Rightarrow R_j < R_k.$$

It is also assumed that there are at least two pricing categories.

Under these assumptions, and under the previously stated assumptions that X and Y, as functions of the time since the last adjustment, are nondecreasing, the price paid for a production lot is a decreasing function of the time between successive adjustments.

The adjustment cost is a decreasing step function of the time between successive adjustments. To see this, consider an example where the length of a production run is 31.5 hr. For an inter-adjustment time greater than 15.75 and less than 31.5 hr, one adjustment is incurred; between 10.5 and 15.75 hr, two adjustments are incurred, etc.

The net revenue is thus a discontinuous function of the time T_a, with the discontinuities corresponding to integer values of T_R/T_a. As T_a increases continuously between these discontinuities, the net revenue decreases. Hence, the maximum net revenue occurs at a value of T_a equal to T_R/n for some positive integer n. (If n is equal to 1, the only adjustment is the initial setup at the beginning of the production run.) Thus, finding the value of T_a yielding the maximum net revenue involves a search amongst the integer values of T_R/T_a.

NUMERICAL EXAMPLES

As a numerical example let Table 1 represent a pricing scheme and

$$N = 630$$

$$T_r = 31.5$$

$$x_0 = 5.0$$

$$y_0 = 0.05$$

$$f_{T_x}(t_x) = 0.5 \exp(-0.5\, t_x)$$

$$f_{T_y}(t_y) = 0.1 \exp(-0.1\, t_x)$$

$$F_{T_x}(t_x) = 1 - \exp(-0.5\, t_x)$$

$$F_{T_y}(t_y) = 1 - \exp(-0.1\, t_y)$$

$$h(t - t_x) = \begin{cases} 0.05\,(t - t_x) & \text{for } T_a \geq t \geq t_x \\ 0 & \text{for } 0 \leq t \leq t_x \end{cases}$$

$$k(t - t_x) = \begin{cases} (t - t_y)^2/100 & \text{for } T_a \geq t \geq t_y \\ 0 & \text{for } 0 \leq t \leq t_y \end{cases}$$

$$A(x)=\begin{cases}(x-5)/0.05 & \text{for } 5<x\leq 5+0.05\times 31.5\\ 0 & \text{for } x\leq 5\end{cases}$$

$$B(y)=\begin{cases}10\sqrt{y-0.05} & \text{for } 0.05<y<0.05+0.01\ (31.5)^2\\ 0 & \text{for } y\leq 0.05\end{cases}$$

$$C_a=6.00.$$

For this example, the probability $P(q_i^*)$ is given by

$$\begin{aligned}(3)\qquad P(q_i^*)=&\int_{L_i}^{U_i}\frac{1}{y_0\sqrt{2\pi}}\exp\left[-(z-x_0)^2/2y_0^2\right]\frac{1-\exp\left[-(0.5+0.1)T_a\right]}{(0.5+0.1)T_a}\,dz\\ &+\int_5^{5+0.05T_a}\int_{L_i}^{U_i}\frac{1}{y_0\sqrt{2\pi}}\exp\left[-(z-x)^2/2y_0^2\right]\frac{0.5\exp\left[-0.1(x-5)/0.05\right]}{0.6T_a}\\ &\left[1-\exp\left\{-0.6T_a-[x-5]/0.05)\right\}\right]\frac{1}{0.05}\,dzdx\\ &+\int_{0.05}^{0.05+T_a^2/100}\int_{L_i}^{U_i}\frac{1}{y\sqrt{2\pi}}\\ &\exp\left[-(z-5)^2/2y^2\right]\frac{0.1\exp\left[-5\sqrt{y-0.05}\right]}{6T_a}\\ &\left[1-\exp\left\{-0.6[T_a-10\sqrt{y-0.05}\,]\right\}\right]\frac{5}{\sqrt{y-0.05}}\,dzdy\\ &+\int_{0.05}^{0.05+T_a^2/100}\int_5^{5+0.5\sqrt{y-0.05}}\int_{L_i}^{U_i}\frac{1}{y\sqrt{2\pi}}\\ &\exp\left[-(z-x)^2/2y^2\right]\frac{0.05}{0.6T_a}\\ &\exp\left\{-0.5[10\sqrt{y-0.05}-(x-5)/0.05]\right\}\\ &-\exp\left\{-0.5\left[T_a-(x-5)/0.05\right]\right.\\ &\left.-0.1[T_a-10\sqrt{y-0.05}]\right\}\frac{1}{0.05}\frac{5}{\sqrt{y-0.05}}\,dzdxdy\\ &+\int_{0.05}^{0.05+T_a^2/100}\int_{5+0.5\sqrt{y-0.05}}^{5+0.05T_a}\int_{L_i}^{U_i}\frac{1}{y\sqrt{2\pi}}\exp\left[-(z-x)^2/2y^2\right]\\ &\frac{0.05}{0.6T_a}\left[\exp\left\{-0.1[(x-5)/0.05-10\sqrt{y-0.05}]\right\}-\exp\left\{-0.5\left[T_a-\frac{(x-5)}{0.05}\right]\right.\right.\\ &\left.\left.-0.1[T_a-10\sqrt{y-0.05}]\right\}\right]\frac{1}{0.05}\frac{5}{\sqrt{y-0.05}}\,dzdxdy.\end{aligned}$$

To calculate the optimum T_a, T_a is first set to 31.5 (the duration of a production run) and the probabilities $P(q_i')$ calculated from Equation (3) for i equal to 1, . . ., 5. These values of $P(q_i')$ are then substituted into Equation (2), and expected revenue is computed. Next an inter-adjustment time corresponding to the next larger integer value of the number of adjustments is used. The process terminates when the next larger number of adjustment intervals gives a lower revenue than did the last.

To find the maximum net revenue, values of T_a corresponding to integer values of the number of adjustments were used to calculate the probabilities $P(q_i')$. These $P(q_i')$ were then used to calculate the net revenue for each T_a used. The results are shown in Table 2. In this problem, the maximum net revenue, \$1,795.20 for inter-adjustment time of 2.86.

TABLE 2. *Sample Computations*

Number of adjustments	T_a	$P(q_1')$	$P(q_2')$	$P(q_3')$	$P(q_4')$	$P(q_5')$	Expected net revenue
1	31.500	0.1419	0.0222	0.2140	0.2108	0.4111	\$241.10
2	15.750	0.0443	0.02533	0.3954	0.3593	0.1756	706.62
3	10.500	0.0158	0.0235	0.5588	0.3555	0.0463	1,061.21
4	7.875	0.0061	0.0212	0.6995	0.2580	0.0152	1,327.50
5	6.300	0.0024	0.0194	0.8077	0.1643	0.0061	1,523.37
6	5.250	0.0010	0.0181	0.8816	0.0962	0.0031	1,653.42
7	4.500	0.0004	0.0173	0.9259	0.0541	0.0022	1,728.51
8	3.9388	0.0002	0.0169	0.9506	0.0304	0.0019	1,767.61
9	3.500	0.0000	0.0167	0.9640	0.0175	0.0017	1,786.17
10	3.150	0.0000	0.0167	0.9714	0.0103	0.0015	1,793.66
11	2.864	0.0000	0.0168	0.9755	0.0063	0.0014	1,795.20
12	2.6250	0.0000	0.0170	0.9778	0.0040	0.0013	1,793.51

REFERENCES

[1] Gibra, Isaac N., "Optimal Control of Processes Subject to Linear Trends," The Journal of Industrial Engineering, Vol. 18, No. 1 (1967).

[2] Girshick, M. A. and Herman, Rubin, "A Bayes Approach to a Quality Control Model," The Annals of Mathematical Statistics, Vol. 23 (1952).

[3] Pritsker, A. Alan B., "The Setting of Maintenance Tolerance Limits," The Journal of Industrial Engineering, Vol. 14 No. 2 (1963).

[4] Roeloffs, Robert, "Acceptance Sampling Plans with Price Differentials," The Journal of Industrial Engineering, Vol. 18, No. 1 (1967).

SCHEDULING WITH PARALLEL PROCESSORS AND LINEAR DELAY COSTS

Kenneth R. Baker
North Carolina State University

and

Alan G. Merten
the University of Michigan

ABSTRACT

This paper deals with the sequencing problem of minimizing linear delay costs with parallel identical processors. The theoretical properties of this m-machine problem are explored, and the problem of determining an optimum scheduling procedure is examined. Properties of the optimum schedule are given as well as the corresponding reductions in the number of schedules that must be evaluated in the search for an optimum. An experimental comparison of scheduling rules is reported; this indicates that although a class of effective heuristics can be identified, their relative behavior is difficult to characterize.

1. INTRODUCTION

Two basic facets of scheduling are the allocation of resources and the sequencing of tasks. In much of the development of scheduling methodology, it has been helpful to simplify the resource structure in order to focus on problems of sequence. Thus in the case of single-machine finite sequencing and in the case of job-shop sequencing, it is usually assumed that each requirement for processing involves a specified, unique resource (Conway, Maxwell, Miller [5]). A first step in treating resource flexibility then is to deal with one resource type and parallel capability. This study deals with the problem of minimizing mean weighted flowtime with parallel machines and with independent nonpreemptable tasks.

The model to be considered involves m machines and n jobs. The machines are identical and each is capable of processing at most one job at a time. The n jobs are independent (that is, no precedence relations exist among them), simultaneously available at time zero, and can each be processed by at most one machine at a time. In addition, job j has associated with it a processing time (denoted p_j), known in advance, and a weighting factor (w_j), reflecting its value or importance.

For job j, the flowtime (F_j) denotes the time spent in the system until completion. The performance measure of interest is mean-weighted flowtime:

$$\bar{F}_w = \frac{\sum_{j=1}^{n} w_j F_j}{\sum_{j=1}^{n} w_j}.$$

Another way of looking at the same problem is to suppose that each job has an associated delay cost per unit time spent in the system (McNaughton [15]). If w_j denotes this unit delay cost, then under

a given schedule the total delay cost accumulated for job j is w_jF_j. The problem of minimizing total delay cost for the set of n jobs is identical to the problem of minimizing $\bar{F}_w$.

For $m = 1$, the well-known result is that $\bar{F}_w$ is minimized by processing the jobs in nondecreasing order of the ratio p_j/w_j. This ordering will be referred to as weighted shortest processing time (WSPT) sequencing. The sequence which maximizes $\bar{F}_w$ is antithetical sequence, weighted longest processing time (WLPT). For $m > 1$, comparable results do not exist, largely due to the presence of a resource allocation problem superimposed upon the sequencing problem.

Models involving $\bar{F}_w$ as a performance measure have been employed in a diverse set of applications, as represented by several of the references (Riesel [22], Merten [17], Bowdon [3], Coffman and Muntz [4], Baker [2], and Grieshop [10]). Some of these sources treat models which explicitly contain parallel processors. In those cases where the discussion is limited to single processor models, it is not difficult to recognize that the parallel processor case is an important and realistic extension of the specialized model. Therefore, the scheduling problem treated in this paper has a very broad spectrum of potential application areas.

2. THEORETICAL RESULTS FOR THE m-MACHINE PROBLEM

The search for the sequence that minimizes mean flowtime and mean waiting time theoretically must consider all possible schedules of the n jobs on the m machines. This search would be based on the relative weights and processing times of the jobs just as it is in the one-machine case. It is possible, however, to reduce the number of schedules that must be investigated because of certain dominance properties, properties of the mean as a performance function, and symmetries. A schedule of the n jobs on the m machines can be viewed as taking place in the following phases:

(1) partition the n jobs into m sets (machines)

(2) order the jobs on each of the m machines

Because of the linear properties of the mean, the mean flowtime for the n jobs on the m machines is the sum of the mean times for each of the m machines. Therefore, to minimize the mean flowtime over all machines, the mean flowtime must be minimized on each machine (Eastman, Even, and Isaacs [7]). Given that a subset of the jobs is assigned to a particular machine, the optimum sequence for this machine corresponds to the WSPT ordering. Therefore, the number of schedules that must be investigated is no greater than the number of ways of assigning n jobs to m machines since, from the one-machine results, WSPT is known to be optimal on a single machine.

The next reduction in the size of the set of schedules to be investigated comes as a result of the observation that an optimum schedule cannot contain an empty machine. For if a schedule were to include an empty machine, a schedule which has a lower mean flowtime can be obtained by taking a job from a machine where there is more than one job and moving it to the previously empty machine (the waiting-time for the job that is moved is reduced from a positive number to zero). The process of eliminating empty machines can be continued until there are no empty ones left.

Finally, since the machines are identical prior to the assignment of the jobs, certain schedules can be ignored since they are indistinguishable from other schedules.

In order to describe sets of schedules, let

Z = set of all possible ways of scheduling n jobs on m machines;

Z_p = set of all possible ways of scheduling n jobs on m machines if the sequence within a machine is ignored (use the WSPT sequence within each machine);

Z_e = set of all possible ways of scheduling n jobs on m machines using WSPT for each machine and excluding the cases where there is one or more unused machines;

Z_m = set of all possible ways of scheduling n jobs on m machines using WSPT, excluding the unused machine cases and ignoring the indistinguishable schedules.

Therefore, $Z \supset Z_p \supset Z_e \supset Z_m$. The problem is now to find $N(A)$, the number of elements in each of these sets A.

Table 1 shows the expressions for the sizes of the sets defined above and gives examples for some small values of n and m. The derivation of these expressions is given in Merten [17]. Some of these results were derived from previous work in combinatorial analysis (Feller [8], Knuth [14], and Abramovitz and Stegun [1]). Even for these small numbers of jobs and machines, it is clearly important to isolate the subset of schedules that contains the sequence that minimizes mean waiting-time.

The following additional results have been shown for the general m-machine problem:

1. It is sufficient to consider schedules in which there is no preemption of jobs (McNaughton [15]).

2. A lower bound $B(m)$, on the optimum solution can be obtained as follows (Eastman, Even, Issacs [7]): Let $B(1)$ denote optimum value of $\bar{F}_w$ for the given job set when $m = 1$ (given by WSPT). Let $B(n)$ denote the optimum value of $\bar{F}_w$ when $m = n$ (given by assigning each job to a different machine, so that $F_j = p_j$). Then

$$B(m) = \max\left\{B(n), \frac{1}{m}B(1) + \frac{m-1}{2m}B(n)\right\}.$$

3. The problem can be formulated as a dynamic programming problem (Held and Karp [11]). This formulation can include the case where the job execution time may differ depending on which machine is used. While the dynamic programming formulation does lead to some reduction in the computation required to find the optimum solution, the procedure is still inadequate for solving large problems.

TABLE 1. *Number of Ways of Sequencing* n *Jobs on* m *Machines as a Function of the Schedule Set*

	$N(Z)$	$N(Z_p)$	$N(Z_e)$	$N(Z_m)$
(n, m)	$n!\binom{n+m-1}{m-1}$	m^n	$\left\{{n \atop m}\right\} m!$	$\left\{{n \atop m}\right\}$
(3, 2)	24	8	6	3
(4, 2)	120	16	14	7
(3, 3)	60	27	6	1
(4, 3)	360	81	36	6

When all weights are equal, the optimal schedule for the m-machine problem can be constructed by arranging the jobs in nondecreasing order of processing time and then assigning the jobs in this order to a machine as soon as one is made available. In practice, this would correspond to the creation of a

single job-queue in which shortest-first priority prevails. Whenever a machine became available, it would be assigned the highest priority job among those remaining in the queue.

Several alternate constructions will yield optimum solutions as well, and it is useful to consider a different viewpoint. As discussed by Conway, Maxwell, and Miller [5, pp. 77–78] an optimal schedule for the equal-weighting problem can be found as follows:

1. Find the jobs with the m longest processing times and assign them in any order to m different machines.
2. Remove the assigned jobs from consideration and repeat step 1 until all jobs are assigned.
3. At each machine, process the jobs in shortest-first sequence.

Similarly, it has been shown, when all the job processing-times are equal, the optimal allocation has the property that the m jobs with the largest weights are in the first positions on the m machines, the jobs with the m largest weights of those remaining are in the second positions on the m machines, and so on until all the jobs have been assigned a position. (Merten [17]).

3. HEURISTIC SCHEDULING PROCEDURES

The foregoing discussion of the equal-weighting problem serves to identify two basic approaches to the more general problem: a *one-at-a-time* job assignment strategy and an *m-at-a-time* job assignment strategy.

Under the one-at-a-time strategy, which is called heuristic H_1, a priority rule is selected in order to form a ranked list of the jobs. The machine with the smallest amount of scheduled processing is then assigned the first job on the list. This step is repeated until all jobs are assigned to machines and then the jobs assigned to each machine are ordered by WSPT. To illustrate how heuristic H_1 works, consider the 10-job set shown in Table 2 and suppose that five machines are available. Also, suppose that the priority rule WLPT is selected to form the initial ranked list (so that the jobs are considered in the order 10, 9, 8, . . ., 2, 1.) At the start, no processing has been assigned to any machine, so the first five jobs on the list are assigned to five different machines (see Table 3). At this stage, the vector of total processing commitments assigned to each machine is (22, 32, 41, 50, 19). Since the minimum occurs for machine 5, the next job (job 5) is assigned to machine 5. The updated vector of machine commitments is (22, 32, 41, 50, 45). Now the minimum occurs for machine 1, and so the next job (job 4) is assigned to machine 1. The details of the procedure are presented in Table 3, and the final sequence generated by this combination of H_1 and WLPT is shown in Figure 1. If WLPT were replaced by some other priority rules for ranking the jobs initially, H_1 might lead to a different schedule, with a different value of $\bar{F}_w$.

TABLE 2. *A 10-Job Data Set, in which the Jobs Are Numbered in WSPT Order*

Job	1	2	3	4	5	6	7	8	9	10
p_j	5	21	16	6	26	19	50	41	32	22
w_j	4	5	3	1	4	2	5	4	3	2
p_j/w_j	1.25	4.2	5.3	6	6.5	9.5	10	10.2	10.7	11

Under the basic m-at-a-time strategy, which is called heuristic H_m, a priority rule is again selected to form a ranked list of the jobs. The first m jobs on the list are assigned to m different machines. The

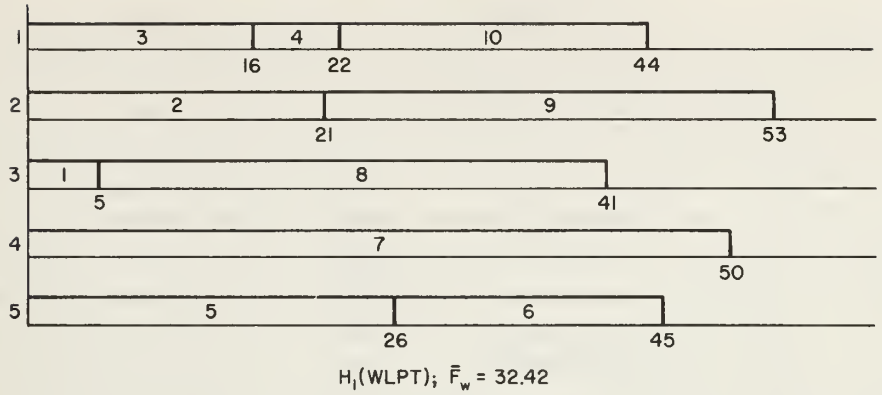


FIGURE 1.

TABLE 3

1. Initial job list {10, 9, 8, 7, 6, 5, 4, 3, 2, 1 }

2. Assignment phase.

Processing commitments	*Job*	*Machine assinged*
(0, 0, 0, 0, 0)	10	1
(22, 0, 0, 0, 0)	9	2
(22, 32, 0, 0, 0)	8	3
(22, 32, 41, 0, 0)	7	4
(22, 32, 41, 50, 0)	6	5
(22, 32, 41, 50, 19)	5	5
(22, 32, 41, 50, 45)	4	1
(28, 32, 41, 50, 45)	3	1
(44, 32, 41, 50, 45)	2	2
(44, 53, 41, 50, 45)	1	3

3. Reorder at each machine by WSPT

Machine	*Sequence*
1	3–4–10
2	2–9
3	1–8
4	7
5	5–6

next m jobs on the list are assigned to m unique machines and so on, until all jobs are assigned. Then WSPT sequencing is applied to each machine. The m-way assignment required at each stage can be specified in more detail. Consider the situation in which the assignment step has been repeated s times, so that $s \cdot m$ jobs are assigned. Taking these assignments to be fixed, consider the subproblem in which it is desired to allocate the next m jobs at stage $s+1$ so that mean weighted flowtime is minimized for all assigned jobs. It is not difficult to show that the optimum allocation in this subproblem is the assignment of the job with the largest weighting factor to the machine with the next smallest processing commitment, and so on. This assignment mechanism is incorporated in H_m. To illustrate this heuristic, consider the example introduced above, and again let WLPT be used to rank the jobs initially. At the first stage, jobs 6–10 are assigned to different machines (see Table 4); this yields a machine commit-

TABLE 4.

1. Initial job list {10, 9, 8, 7, 6, 5, 4, 3, 2, 1}

2. Assignment phase

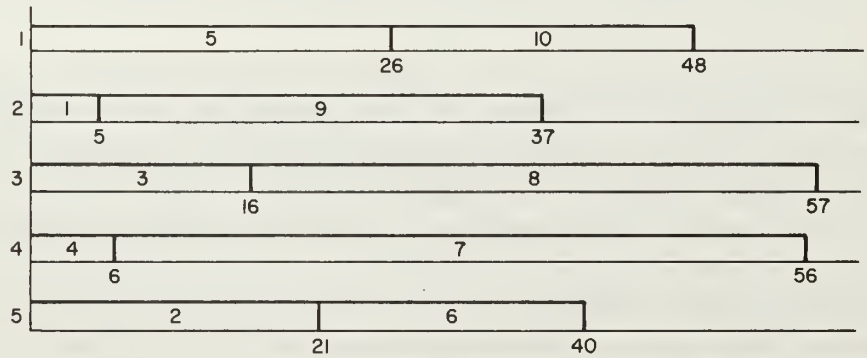
Stage	*Processing commitments*	*Jobs*(w_j)	*Machine assigned*
1	(0, 0, 0, 0, 0)	10 (2)	1
		9 (3)	2
		8 (4)	3
		7 (5)	4
		6 (2)	5
2	(22, 32, 41, 50, 19)	5 (4)	1
		4 (1)	4
		3 (3)	3
		2 (5)	5
		1 (4)	2

3. Reorder at each machine by WSPT

Machine	*Sequence*
1	5–10
2	1–9
3	3–8
4	4–7
5	2–6

ments vector of (22, 32, 41, 50, 19). At the second stage the machines are ordered smallest-first by this commitment (5–1–2–3–4) and the jobs are ordered largest-first by weighting factor, ties being broken arbitrarily (2–5–1–3–4). This leads to the assignment of job 2 to machine 5, job 5 to machine 1, job 1 to machine 2, job 3 to machine 3, and job 4 to machine 4. The details of this heuristic are given in Table 4 and Figure 2. Once again, the use of a priority rule different from WLPT might lead to a different schedule. In any case, when n is an even multiple of m (as in the example problem), heuristic H_m will always assign the same number of jobs to every machine.

One variation of this form of H_m is to relax the restriction that the m jobs must be assigned to different machines at each stage. In this case, a possible heuristic is to treat the m jobs in decreasing order of their weighting factors and to assign them one at a time to the machine with the smallest processing



H_m(WLPT); $\bar{F}_w$ = 32.67

FIGURE 2.

commitment. Under this heuristic, called H_x, it is possible that several of the m jobs considered at a given stage will be assigned to the same machine. We were surprised, however, to find in our experimentation that H_x was consistently less effective than the other two heuristics.

The priority rule at the heart of each heuristic can be selected from a variety of orderings which are potentially effective in sequencing. At least five basic priority rules are of interest: shortest processing time (SPT), longest processing time (LPT), their weighted versions (WSPT and WLPT), and largest weighting factor (W). Used in conjunction with the three heuristics H_1, H_m, and H_x, they yield 15 distinct scheduling procedures with various combinations being denoted as H_1 (WLPT), H_m (W), etc. The remainder of the discussion deals with the solutions generated by these procedures.

For the 10-job example of Table 2, an examination of the 15 scheduling procedures reveals that the best schedule is not one of those in Figures 1–3, but is one produced by H_m (WSPT), with $\bar{F}_w = 32.30$. (In order to determine an optimum schedule, it would be necessary to examine $N(Z_m) = 42{,}525$ schedules in this problem.) The second best schedule is produced by H_1 (WLPT) and the third best by H_1 (LPT). It is interesting to observe, however, that for the same job set with a different number of machines, the relative ordering of the scheduling procedures is somewhat different. Table 5 displays results for $2 \leq m \leq 6$. Two important properties are evident: first, several different procedures produce best schedules at least once in the five different problems; and second, the set of three best procedures is different for every value of m. There is no clear-cut choice for the best scheduling rule, nor is there yet even a convincing choice between heuristics H_1 and H_m.

This instability of relative performance among scheduling procedures might well be particularly characteristic of small problems. When n is small, a change in the scheduling of one or two jobs can represent a significant change in the overall performance measure, whereas this is much less likely to be the case when n is large. As illustrated by this example problem, when the job set is small there may be considerable nonuniformity in the effectiveness of a particular rule. It is doubtful that a truly optimum procedure will exist among the heuristic procedures examined here.

To shed some light on the question of the effect of problem size, a more detailed investigation of larger problems was carried out.

TABLE 5. *Rankings of the Three Best Rules for Different Values of* m.

m	2	3	4	5	6
Best	H_1 (WSPT)[a]	H_1 (LPT)	H_1 (WSPT)	H_m (WSPT)	H_m (W)[a]
Second	H_m (WSPT)[a]	H_m (WSPT)	H_1 (WLPT)	H_1 (WLPT)	H_1 (W)[a]
Third	H_x (WSPT)	H_1 (SPT)	H_m (W)	H_1 (LPT)	H_1 (WSPT)[a]

[a] Ties.

4. EXPERIMENTS WITH LARGE JOB SETS

Six large job sets, of size $n = 100$ jobs, were constructed for further experimentation. The job sets were generated as follows: (a) the processing times in each set consisted of random samples drawn from a distribution with a mean of 50, (b) the weights in each set were independent of the processing

times and were samples drawn from a distribution with a mean of five. The job sets were distinguished by the forms of the distribution, as shown in Table 6. For each job set, the same jobs were scheduled for m parallel machines, where m was again varied from 2 to 6. With six data sets and five versions of parallelism, 30 different problems were posed, and each of the 15 scheduling procedures was tested on each of the 30 problems.

Detailed results for data set four are displayed in Table 7. For each combination of scheduling procedure and number of machines, the schedule value is given as the difference between $\bar{F}_w$ and the lower bound, along with its ranking among the 15 procedures. This particular data set is perhaps a typical among the six tested, but it does serve to highlight many of the characteristics of the problem. Specifically, the results show that there is only a limited amount of dependence in the rankings as m is varied. Although the five different values of m do not generate completely independent sets of observations, they do convey much more information than the results for any single value of m alone.

Three different rules emerged as best in this job set: H_m (WSPT), H_1 (WLPT), and H_1 (WSPT). Thus two different heuristics H_1 and H_m emerged as best. This configuration dramatically illustrates

TABLE 6

Data Set	1	2	3	4	5	6
p-distribution	$U(0, 100)$	$N(50, 10)$	$U(0, 100)$	$N(50, 10)$	$U(0, 100)$	$X(50)$
w-distribution	$U(0, 10)$	$U(0, 10)$	$N(5, 1)$	$N(5, 1)$	$X(5)$	$X(5)$

Notation: $U(a, b)$ Uniform on the interval a to b.
$N(a, b)$ Normal with mean a and standard deviation b.
$X(a)$ Exponential with mean a.

TABLE 7

m	2		3		4		5		6	
H_1 (SPT)	0.65	8	1.23	9	1.91	9	2.39	9	1.55	5
(WSPT)	0.10	2	0.18	2	0.18	1	0.31	2	0.29	1
(LPT)	2.04	11	1.32	10	1.55	8	1.90	7	2.43	11
(WLPT)	0.12	4	0.16	1	0.19	2	0.35	4	0.33	2
(W)	0.51	7	3.02	13	2.18	11	2.17	8	1.94	8
H_m (SPT)	20.89	14	15.91	14	13.91	14	12.07	15	11.98	14
(WSPT)	0.09	1	0.22	3	0.22	3	0.27	1	0.38	3
(LPT)	21.53	15	16.40	15	14.62	15	11.18	14	12.80	15
(WLPT)	0.11	3	0.23	4	0.27	4	0.34	3	0.17	4
(W)	2.12	12	0.71	7	1.44	6	2.45	10	1.94	8
H_x (SPT)	5.88	13	2.27	11	2.92	12	3.92	13	4.29	13
(WSPT)	0.36	6	0.31	5	1.48	7	1.02	6	2.21	10
(LPT)	0.98	9	1.22	8	4.33	13	2.64	11	3.14	12
(WLPT)	0.27	5	0.32	6	0.57	5	0.88	5	1.85	7
(W)	1.22	10	2.74	12	1.96	10	2.79	12	1.68	6

The numbers shown for each combination are (1) the difference between $\bar{F}_w$ and the lower bound and (2) the rank of the $\bar{F}_w$ value among the rules tested.

that no single heuristic will always be associated with the best schedule and that no single priority rule will always be associated with the best schedule.

Secondly, the rankings indicated quite clearly that the weighted priorities are more effective than their unweighted counterparts. Only under H_x did a weighted priority lead to a rank below sixth.

Thirdly, H_x did not produce a schedule ranked better than fifth and was clearly worse than the other heuristics. Presumably, H_x suffers from the fact that it does not necessarily distribute the jobs in equal numbers among machines.

The heuristic H_m, by contrast, is restricted to distributing the jobs in equal numbers among machines. While this characteristic is sometimes favorable, it is distinctly unfavorable in the case of the unweighted priorities, which ranked 14 and 15 in all six problems.

Finally, there appears to be no overall clear choice between the priorities WSPT and WLPT. For H_m, WSPT seems to be uniformly more effective, but for H_1 no similar conclusion can be drawn. In some respects, this may be the most surprising property illustrated by these results, for although WLPT maximizes $\bar{F}_w$ for $m=1$, it can be incorporated into the parallel processor case in a desirable way.

The important results in the six data sets are summarized in Table 8 by the use of rankings where the specific rules which produced the three best schedules are shown in ranked order for all 30 problems. Of the 30 outcomes, the distribution of best schedules was as follows:

	Best	*Second*	*Third*	*Total*
H_1 (WSPT)	21	9	0	30
H_m (WSPT	6	5	17	28
H_1 (WLPT)	3	16	8	27

TABLE 8. *Comparison of Rules*

$m=$	2	3	4	5	6
*DS*1	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)
*DS*2	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WSPT) H_m (WSPT) H_1 (WLPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WSPT) H_m (WSPT) H_1 (WLPT)
*DS*3	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WSPT) H_m (WSPT) H_1 (WLPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_1 (WLPT) H_1 (WSPT) H_m (WLPT)
*DS*4	H_m (WSPT) H_1 (WSPT) H_m (WLPT)	H_1 (WLPT) H_1 (WSPT) H_m (WSPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_m (WSPT) H_1 (WSPT) H_m (WLPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)
*DS*5	H_m (WSPT) H_1 (WSPT) H_m (WLPT)	H_1 (WLPT) H_1 (WSPT) H_m (WSPT)	H_1 (WSPT) H_m (WSPT) H_1 (WLPT)	H_1 (WSPT) H_m (WSPT) H_1 (WLPT)	H_1 (WSPT) H_m (WSPT) H_m (WLPT)
*DS*6	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_m (WSPT) H_1 (WSPT) H_1 (WLPT)	H_1 (WSPT) H_1 (WLPT) H_m (WSPT)	H_m (WSPT) H_1 (WSPT) H_1 (WLPT)	H_m (WSPT) H_1 (WSPT) H_1 (WLPT)

H_1 (WSPT) most frequently produced the best schedule and always produced one of the two best schedules. H_m (WSPT) and H_1 (WLPT) were less likely to produce the best schedule, but nearly as likely to produce one of the three best. While the size of the problem precludes a comparison of the best heuristic solution with the true optimum, we observed that H_1 (WSPT) was within 1 percent of the lower bound 95 percent of the time. Therefore, it appears that only very slight improvements could possibly be made over the solution obtained with this heuristic procedure.

5. CONCLUSIONS

If an optimum rule for this problem exists (that is, a scheduling mechanism more efficient than enumeration) it is likely to be quite complicated. Furthermore, extensions to flowtime problems with multiple resource types or with nonstatic job arrivals would also appear to be complex.

The primary element in attempting to minimize $\bar{F}_w$ with parallel processors is the use of the condition that WSPT should prevail for each processor. This condition is so important to near-optimal scheduling that only marginal improvements can be expected from sophisticated assignments of jobs to machines. Moreover, this investigation suggests that the relative behavior of heuristic procedures for this assignment process may be extremely difficult to characterize in general. The prospect is that special problem attributes (distribution of processing times, number of machines, etc.) will affect the performance of different procedures and perhaps render the concept of an "optimum rule" meaningless.

The results for the large job sets indicate that H_x is the least effective of the three heuristics tested and that neither H_1 nor H_m is consistently best. Indeed, for a given set of jobs, it is possible that the relative performance of H_1 and H_m is reversed as m is varied. Nevertheless, H_1 did appear to be perceptibly more likely than H_m to produce the best schedule. The outcome is a pleasant surprise in that H_1 is the simplest of the three heuristics to implement and H_x is the most difficult.

In much the same way, no priority ordering was consistently best, although it was clear that weighted priorites were more reliable than unweighted priorities. The effectiveness of WLPT might be attributable to the fact that longest-first sequencing tends to distribute processing fairly equally among machines, as discussed by Kedia [13] and illustrated in Figure 1. Nevertheless, WSPT appeared to be the best priority ordering.

The fact that both WSPT and WLPT were effective might suggest that weighting factors are more important job traits than processing times. Yet the largest-weight priority was unable to produce one of the three best schedules in any of the 30 problems.

With regard to the effect of processing time distributions and weighting factor distributions, the results are inconclusive. If anything, H_1 (WSPT) was most effective when the weights were uniformly distributed and was least effective for data sets 4 and 6, but these represented the job sets with the least and most variability. More testing would be required to determine whether there is a significant distribution effect. From the limited scope of these results, however, one might infer that the conclusions hold for a wide variety of distributions.

Granting the lack of consistency which is inherent in the problem, the data in Table 8 certainly recommend H_1 (WSPT) as the most effective scheduling procedure. In addition to the high frequency with which it produced good schedules, H_1 (WSPT) has other advantages. First, it is a logical rule to use, since it is a generalization of the optimum rule for the multiprocessor problem with equal weights. Secondly, it is a one-pass procedure, and does not require a reordering of the jobs once they have been assigned to machines. It is slightly simpler than H_m (WSPT), which includes an additional assignment mechanism at each stage, and is probably the simplest procedure of those studied. Finally, H_1 (WSPT)

structurally is a dispatching procedure: the final job assignments are made at chronologically ordered points in time (i.e., in the order they would be implemented.) This type of structure is likely to be more adaptable as part of a larger, more complex problem than two-pass procedures or iterative schemes. In particular, problems with multiple resource types or with dynamic job arrivals are important extensions of the problem considered here, and they can accomodate the H_1 (WSPT) heuristic without major obstacles.

BIBLIOGRAPHY

[1] Abramowitz, M. and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series, (1964), pp. 824–825.

[2] Baker, N. R., "Optimal User Search Sequences and Implications for Information System Operation," American Documentation, Vol. 20, No. 3. (July 1969), pp. 203–212.

[3] Bowdon, E. K., "Priority Assignment in a Network of Computer," IEEE Transactions on Computers, C–18, 11 (Nov. 1969), pp. 1021–26.

[4] Coffman, E. G. and R. R. Muntz, "Models of Pure Time-Sharing Disciplines for Resource Allocations," Proc. 24th ACM National Conference (1969), pp. 217–228.

[5] Conway, R. W., W. L. Maxwell and L. W. Miller, *Theory of Scheduling* (Addison-Wesley, Reading, Mass., 1970).

[6] Denby, D. C., "Minimum Downtime as a Function of Reliability and Priority Assignments in Component Repair," Journal of Industrial Engineering Vol. 17, No. 7 (1967), pp. 436–439.

[7] Eastman, W. L., S. Even and I. M. Isaacs, "Bounds for the Optimal Scheduling of n Jobs on m Processors," Management Science Vol. 11, No. 2 (Nov. 1964), pp. 268–279.

[8] Feller, W. "An Introduction to Probability Theory and Its Applications" (John Wiley & Sons, New York, 1957), Vol. 1.

[9] Gapp, W., P. S. Mankekar and L. G. Mitten, "Sequencing Operations to Minimize In-process Inventory Costs," Management Science Vol. 11, No. 3 (Jan. 1965), 476–484.

[10] Grieshop, D. S. "An Analysis of the System Effectiveness of a Sequential Manpower Training Model," M.S. Thesis, School of Industrial and Systems Engineering, Georgia Institute of Technology (Jan. 1972).

[11] Held, M. and R. M. Karp, "A Dynamic Programmic Approach to Sequencing Problems," Journal of SIAM Vol. 10, No. 1 (Mar. 1962), pp. 196–210.

[12] Hu, T. C. "Parallel Sequencing and Assembly Line Problems," Operations Research Vol. 9, No. 6 (Nov. 1961), pp. 841–848.

[13] Kedia, S. K., "A Job Shop Scheduling Problem with Parallel Machines," Technical Report 71–6, Department of Industrial Engineering, University of Michigan (1971).

[14] Knuth, D. E., *The Art of Computer Programming: Fundamental Algorithms* (Addison-Wesley, Reading, Massachusetts, 1968) pp. 51–73.

[15] McNaughton, R., "Scheduling with Deadlines and Loss Functions," Management Science, Vol. 6, No. 1 (Oct. 1959), pp. 1–12.

[16] Maxwell, W. L., "On the Generality of the Equation $L=\lambda W$," Operations Research Vol. 18, No. 1 (Jan.–Feb. 1970), pp. 172–174.

[17] Merten, A. G., "Some Quantitative Techniques for File Organizations," Technical Report No. 15, University of Wisconsin Computing Center (June 1970).

[18] Mitten, L. G., "An Analytic Solution to the Least-Cost Testing Sequence Problem," Journal of Industrial Engineering, Vol. 11, No. 1 (Jan.–Feb. 1960), p. 17.

[19] Muntz, R. and E. Coffman, "Optimal Pre-emptive Scheduling on Two-Processor Systems," IEEE Transactions on Computers, Vol. C–18, No. 11 (Nov. 1969), pp. 1017–1020.

[20] Price, H. W., "Least-Cost Testing Sequence," Journal of Industrial Engineering, Vol. 10, No. 4, (July–Aug. 1959), pp. 278–279.

[21] Rau, J. G., "Minimizing a Function of Permutations of n Integers," Operations Research, Vol. 19, No. 1, (Jan.–Feb. 1971), pp. 237–240.

[22] Riesel, H. "In which Order are Different Conditions to be Examined?," *BIT*, Vol. 3 (1963), pp. 255–256.

[23] Rothkopf, M., "Scheduling Independent Tasks on Parallel Processors," Management Science, Vol. 12, No. 5 (Jan. 1966), pp. 437–447.

[24] Smith, W. E., "Various Optimizers for Single-Stage Production," Nav. Res. Log. Quart. *3*, 59–66 (1956).

NEWS AND MEMORANDA

Logistics Research Conference

The Office of Naval Research and the George Washington University, with the cooperation of the Air Force Office of Scientific Research and the Army Research Office, announce a Logistics Research Conference to be held at the George Washington University, Washington, D.C., on 8–10 May 1974. The main objectives of the Conference are to survey major developments and difficulties in government, industrial and military logistics research and applications since World War II, and to assess outstanding current problems and promising new research techniques.

Areas of research activity have been categorized, as follows: 1) applications of mathematical programming; 2) applied case studies; 3) design of systems; 4) inventory systems; 5) data collection, representation, and analysis; 6) measurement of performance; 7) probabilistic methods; 8) production and procurement; 9) reliability, maintainability, and availability; 10) simulation; 11) statistical methods; and 12) transportation and scheduling.

Contributed papers are welcome. Abstracts and inquiries may be addressed to Ms. Henrietta Jones, Department of Operations Research, The George Washington University, Washington, DC 20006. (Phone: 202/676–7504.) Further information may also be obtained from Professors Anthony V. Fiacco (202/676–7511), W. H. Marlow (202/676–7503), or Henry Solomon (202/676–7521) at the University, or from Mr. Marvin Denicoff (202/692–4304) at the Office of Naval Research.

INDEX TO VOLUME 20

CUMULATIVE INDEX FOR VOLUMES 1-20

U.S. GOVERNMENT PRINTING OFFICE: 1973—541-388:2

INFORMATION FOR CONTRIBUTORS

The NAVAL RESEARCH LOGISTICS QUARTERLY is devoted to the dissemination of scientific information in logistics and will publish research and expository papers, including those in certain areas of mathematics, statistics, and economics, relevant to the over-all effort to improve the efficiency and effectiveness of logistics operations.

Manuscripts and other items for publication should be sent to The Managing Editor, NAVAL RESEARCH LOGISTICS QUARTERLY, Office of Naval Research, Arlington, Va. 22217. Each manuscript which is considered to be suitable material for the QUARTERLY is sent to one or more referees.

Manuscripts submitted for publication should be typewritten, double-spaced, and the author should retain a copy. Refereeing may be expedited if an extra copy of the manuscript is submitted with the original.

A short abstract (not over 400 words) should accompany each manuscript. This will appear at the head of the published paper in the QUARTERLY.

There is no authorization for compensation to authors for papers which have been accepted for publication. Authors will receive 250 reprints of their published papers.

Readers are invited to submit to the Managing Editor items of general interest in the field of logistics, for possible publication in the NEWS AND MEMORANDA or NOTES sections of the QUARTERLY.

NAVAL RESEARCH
LOGISTICS
QUARTERLY

DECEMBER 1973
VOL. 20, NO. 4
NAVSO P-1278

CONTENTS

ARTICLES

OFFICE OF NAVAL RESEARCH
Arlington, Va. 22217